On the Causal Structure of Quantum Space-Time

Spin-foams as Quantum Causal Histories

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A meus pais.

Eles não sabem que o sonho
É uma constante da vida
Tão concreta e definida
Como outra coisa qualquer
[...]

In *Movimento Perpétuo*, 1956

Abstract

This work focuses on the framework of spin-foam models and their possible causal structure. In this context we propose a prescription for identifying a spin-foam model as a quantum causal history. To do so we first review the general construction of spin-foam models, with particular emphasis on the EPRL model, and we study the role of causal loops in quantum mechanical evolution. We further argue that the EPRL-type models do not admit such a quantum causal history description, as we show that the quantum operators they induce fail to be unitary.

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Contents

Introduction

Quantum mechanics (in its quantum field theory guise) is widely regarded as the most well-tested physical theory we currently possess. And general relativity, while not so extensively tested, seems to be equally well-established as a fundamental theory of nature. In the absence of strong experimental evidence of the failure of either of these theories at some high energy regime, why do we find it pertinent to study the problem of a possible unification of both quantum mechanics and gravity, specially when such a unification is theorized to manifest itself mainly at the Plank scale $l_P \sim 10^{-35}$m, so far away from the scope of our instruments?

Although it is true that the quantum gravity program is not motivated by a wealth of physical evidence (as once happened with the formulation of quantum mechanics, necessary at the time due to the persistent insistence of experiments not to behave as expected), it also happens that the world-views afforded by both conceptions of nature seem to be unconciliatable: while general relativity tells the tale of a world with smooth structures and fully deterministic systems, where time and space are dynamical physical objects like any other, quantum mechanics describes a probabilistic one where time and space are but parameters. Given this state of affairs it would seem that nature has forced our hand in making progress; that in the absence of new experimental data, caused by our own "excessively good" ability at constructing physical models, new progress must be made through the hope that the world admits a conceptually consistent understanding. The approach we take is then an optimistic one: that in the pursuit of a conceptual and theoretical unification, new verifiable predictions might come up.

This work is therefore carried out in the context of the quantum gravity program. Our framework of choice is the spin-foam approach, which is a manifestly non-perturbative proposal for assigning amplitudes to quantum states associated with geometry. These models can be derived from continuum classical theories, and they allow for a conceptualization of spacetime as an arrangement of fundamental "atoms", such that the quantum dynamics of spacetime is encoded in each fundamental element. We are mainly interested in understanding if a notion of causality is implemented at the quantum level in such objects, and we pursue a possible correspondence with the framework of quantum causal histories, another proposal for a quantum gravity framework that explicitly encodes a causal structure.

The organization of this document is as follows. In Chapter 1 we review the theory of general relativity and related models, focusing both on the formal aspects and on the conceptual ones. In particular we discuss the tetrad formulation of gravity, which proved to be very useful in many quantum gravity theories, as well as the constrained topological gravity model, used by most spin-foam formulations. In Chapter 2 we turn towards a comprehensive overview of spin-foam models, discussing not only their structure but also the physical and mathematical motivations behind them, focusing mainly on the famous EPRL spin-foam. Chapter 3 is dedicated to a discussion on causality. We introduce the framework of causal sets and quantum causal histories, and study a criterion on quantum operators for linear evolution in the presence of causal loops proposed initially by Deutsch. By defining an operator from which spin-foam amplitudes can be extracted, we establish a prescription for identifying a spin-foam as a quantum causal history. We moreover show such an operator to be non-unitary for a large class of models of the EPRL-type. Finally, to accompany the physical discussion in this work, we have also included a number of appendices containing some of the relevant mathematical tools.

Chapter 1

The Classical Theory of Space and Time

1.1 The emancipation of spacetime

This section is dedicated to a succinct discussion of the theory of general relativity in its standard formulation, as a preparation for more advanced topics. We start with a brief discussion of the physical principles of the theory and the framework on which it stands. Some of the content of the theory, in particular its conceptual underpinnings and the observables it suggests, is also reviewed.

1.1.1 Physical principles of the theory

General relativity frequently holds a special place in the family of physical theories that we currently possess. This has much to do with how the theory was motivated and constructed *ab-initio*: not necessarily as a model designed *ad-hoc* to explain some unexpected experimental results, but as a structured consequence of a set of principles we hold to be true about the world. In this sense it incorporates in its content a long history of insights on the inner-workings of nature, and still today it surprises us with its implications, not always intuitive or expectable. As the very first step towards the discussion of a possible quantum extension down the line, we start here by recalling the postulates of the theory.

General relativity hinges on a pair of basic principles, discussed prominently by Einstein in [1]:

- General Principle of Relativity: *the content of physical laws should not depend on the reference frame used to describe them.*
- Principle of Equivalence: *an inertial reference frame subject to gravity is indistinguishable from an accelerated one.*

The *general principle of relativity*, as a guiding postulate, is not exclusive to the theory of general relativity. In fact it demands that physical theories, whatever they refer to, must be formulated in such a way that their content, *i.e.* their predictive value, does not depend on the particular reference frame one considers them in. To this end the theory of smooth bundles and their sections is most adequate, for it allows the description of objects without necessary reference to a frame. Although those objects do admit a coordinate description, there is no single preferred reference for those coordinates, and the object can be described equivalently in any other frame. In this way the laws of physics become coordinate-independent and truly universal. We expect a physical theory satisfying the *general principle of relativity* to be then formulated as an action functional over some smooth spacetime manifold M,

$$S = \int_M \mathcal{L}\,, \tag{1.1}$$

for some chosen lagrangian density $\mathcal{L}$. In this sense, any sensible physical theory must be a theory of fields on a spacetime manifold, and such is the case for, as a strong example, any modern quantum field theory.

For the particular case of general relativity, which concerns itself with the problem of describing gravitation, it is the second *principle of equivalence* that allows one to determine concretely the appropriate degrees of freedom of the theory. It follows from recognizing that the inertial mass, associated to the fundamental law of acceleration $F = ma$, is perfectly equivalent to the gravitational mass, as the "charge" of the gravitational force. Under such an equivalence there is no empirical analysis that can discern between an accelerated reference frame and an inertial one in the presence of a gravitational field (as Einstein's elevator is a famous example of). One may then try to establish gravitation as a theory of non-inertial frames. From special relativity, Einstein had already argued that to each inertial frame a Minkowski metric η should be associated, responsible for identifying those transformations that would leave the frame inertial as the isometry group of the metric. These are the usual Lorentz transformations Λ, such that, given an inertial frame $\mathcal{F}$ with metric η, the frame transforms into another one through $\mathcal{F} \to \Lambda\mathcal{F}$, $\eta \to \Lambda\eta\Lambda^T = \eta$, and the metric is left invariant. But one can consider other transformations that take an inertial frame to a non-inertial one, $\mathcal{F} \to T\mathcal{F}$, and such a transformation $\eta \to g \neq \eta$ would not leave the Minkowski metric invariant. It is therefore natural to associate the character of inertia of frames to a particular choice of a metric, in a way that to each non-inertial frame, and to each metric, there corresponds only one possible transformation, up to a Lorentz one, that takes it into a Minkowski form. Hence, the *principle of equivalence* suggests a description of gravity in terms of a Lorentzian metric over spacetime, and we arrive at the theory

$$\mathcal{L} = \mathcal{L}(g, \phi_M)\,, \tag{1.2}$$

where ϕ_M generically represents matter fields, and the concrete form of the lagrangian can then be determined through further physical arguments, as was done by Einstein.

1.1.2 General relativity in a nutshell

The theory of general relativity is the theory of a 4-dimensional smooth Lorentzian manifold (M, g), considered to be *spacetime*, equipped with the unique Levi-Civita connection ∇. The manifold is endowed with an atlas $\mathcal{A}$, and given some chart $\phi : U \subset M \to \mathbb{R}^{3,1}$ the connection acts on a local section $X : U \to M$, with $X = X^\mu \partial_\mu$, as

$$\nabla_\mu X^\nu = \partial_\mu X^\nu + \Gamma^\nu_{\mu\alpha} X^\alpha \,, \tag{1.3}$$

where $\Gamma^\nu_{\mu\alpha}$ are the connection coefficients.

The metric g is taken to be the fundamental physical field of the theory, which is described by the Einstein-Hilbert action (up to boundary terms)

$$S_{EH} = \int_M \mathrm{d}^4x \sqrt{-g}\,(R - 2\Lambda + 2\kappa \mathcal{L}_M)\,, \tag{1.4}$$

where R is the Ricci scalar of the connection, Λ the cosmological constant and $\mathcal{L}_M$ the matter lagrangian. Variation of the action with respect to the metric results in the Einstein field equations [2],

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa T_{\mu\nu}\,, \tag{1.5}$$

where $R_{\mu\nu}$ are the components of the Ricci curvature and $T_{\mu\nu} = \frac{-2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}_M)}{\delta g^{\mu\nu}}$ is the energy-momentum tensor. Operationally, the equations (1.5) are always solved in coordinates, and as such they only determine the metric on some open set U of the manifold. One then has to use physical arguments to specify the concrete manifold M one is considering, and then patch the solutions for the metric over an open covering $\{U_i\}$ of the manifold.

Having determined a concrete Lorentzian metric on M, general relativity postulates that systems propagate in spacetime along generalized "straight" paths, obtained by parallel transporting a vector by the connection,

$$\frac{\mathrm{d}^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta} \frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau} \frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} = 0 \tag{1.6}$$

where τ parametrizes the curve $\gamma : (0,1) \to M$, and $x^\mu = (\phi \circ \gamma)^\mu$ are the components of the curve in the chart. Equivalently, one may write $\nabla_{\dot\gamma}\dot\gamma = 0$, where $\dot\gamma$ denotes the tangent vector field to the curve.

1.1.3 Measurements in spacetime

To understand the content of the theory of general relativity one needs to consider its predictive power; the statements it is able to make about the world. The abstract formalism described in the previous subsection is nothing more than a theoretical construction: the symbols must now be matched to natural objects which can be measured.

In field theories over a Minkowski space-time the manifold under consideration is simply $\mathbb{R}^{3,1}$, which admits a single chart to itself. One then has global coordinates available with which to describe the degrees of freedom of the theory. A set of rods and clocks is associated to these coordinates, in such a way that one can measure some initial conditions $\phi(x_0, t_0)$ and solve the equations to find the system at some later time $\phi(x, t)$. As long as the points (x_0, t_0) and (x, t) are measured with the same device, the predicted state of the system at later times should match the experiment.

The situation in general relativity is considerably more difficult (on this, see *e.g.* [3, 4, 5, 6] and references therein), because the abstract mathematical charts do not have any intrinsic physical meaning. Consider a universe (M, g) and a curve $\gamma : (0,1) \to M$ parametrized by $\tau \in (0,1)$. There exists a natural coordinate-independent quantity, the proper length L of the curve,

$$L_\gamma = \int_\gamma \mathrm{d}\tau \sqrt{g(\dot\gamma, \dot\gamma)}\,, \tag{1.7}$$

where $\dot\gamma$ denotes the tangent vector field to γ. This quantity is a generalization of the proper-time in special relativity, and it is most natural to consider it to be what an ideal clock traveling through γ would measure: the "aging" of the system. It is important to note that this quantity depends non-trivially on the dynamics of the metric, and already here one understands why in general relativity one cannot take the charts of the atlas of M to correspond to physical reference frames: under a chart $\phi : U \subset M \to \mathbb{R}^{3,1}$, the quantity L_γ does not simply correspond to $(\phi \circ \gamma(0))^0 - (\phi \circ \gamma(1))^0$; in general relativity, coordinate charts *do not* correspond to laboratory frames, except in circumstantial approximations. As Wigner puts it [3],

"*[...] to some degree we mislead both our students and ourselves when we calculate [...] the mercury perihelion motion without explaining how our coordinate system is fixed in space [...].*"

How then should lengths and time intervals be measured in general relativity? As hinted at in the previous subsection, physical systems are described in general relativity by curves and fields in M. To make measurements we certainly need a clock, so let us populate our universe with a point-like system $\xi : (0,1) \to M$ carrying a clock $T : M \to \mathbb{R}$. The clock field associates to each point in M a number in $\mathbb{R}$, like the numbers on a digital stopwatch, and we may demand that it is an ideal clock by requiring that

$$T(\xi(\tau_2)) - T(\xi(\tau_1)) = \int_{\tau_1}^{\tau_2} \mathrm{d}\tau \sqrt{g(\dot\xi, \dot\xi)}\,, \tag{1.8}$$

for any $\tau_1, \tau_2 \in (0,1)$ (we shall not concern ourselves with how such a clock could be constructed, but we assume it can. Even a pendulum clock at rest on the surface of the earth is within a good approximation ideal, as discussed in [7]). Notice that while both T and ξ can be described in a chart through $T \circ \phi^{-1}$ and $\phi \circ \xi$, the quantity $(T \circ \phi^{-1}) \circ (\phi \circ \xi) = T(\xi)$ is independent of any coordinates, and as such it contains meaningful physical information: the time measured by a clock carried by the system.

To measure lengths, we can further add to our universe the world-line of a photon $\gamma : (0,1) \to M$ and some mirror, stationary relative to the clock. We can think of the photon as being emitted by the system ξ, reflected on the mirror, and detected again by the system. The curve described by the photon will then intersect the world-line of the system in two points, and a distance can be computed by multiplying the time interval measured by the clock as in equation (1.7) by the speed of light.

Predictions in general relativity are then made by solving the pertinent equations in some arbitrary chart, and then relating the quantities in a coordinate-independent manner. In this way, for example, the Ricci curvature at some coordinates $R(x^\mu)$ does not have any physical content, neither does the representation of the curve $\xi^\mu(\tau)$ in components, but the quantity $R(\xi(\tau))$, representing the scalar curvature at some proper time of the system ξ, does. Of course, this means that in gravity the rods and clocks with which we make empirical measurements are also dynamical, and must be incorporated in the theory itself, and this makes general relativity a hard theory to work with.

1.1.4 Conceptual implications

Having established the structural skeleton of the theory of gravity, and discussed its founding principles, a reflection on what the physical implications of such a theory is in order: what is the theory telling us?

- *Time and space as physical entities*

  As previously discussed, general relativity is formulated over a smooth Lorentzian manifold (M, g). The inclusion of the metric in the description of spacetime implies that distances and time intervals will not be simply coordinate differences, but complicated functions of the metric as in equation (1.7). Since the metric is itself dynamical, and dependent on the matter distribution, the operationally meaningful notions of time and space, as measured by an observer, are also dynamical. In this manner space and time are no longer constituents of a fixed background stage on which everything else moves, but rather they are by themselves full-fledged physical entities with their own dynamics. Through general relativity, space and time emancipate themselves from the Newtonian viewpoint of absolute structures.

- *No background structure*

  From the previous point it follows that there is no fixed background on which matter and energy exist. While in the special theory of relativity such a structure is present in a global Minkowski metric, securing even the possibility of a globally-defined chart, in general relativity there exists no such *a priori* structure. Since lengths and time intervals are determined dynamically, rather than being associated to some fixed structure, even the base manifold on which the theory is formulated (usually considered to be space-time itself) can be argued to be more of a parametrization mechanism than actually space-time. Moreover, the equations of motion of general relativity refer exclusively to the metric field and the energy-momentum tensor in local coordinates, making no principle statement on what the manifold should even be: it simply needs to be able to accommodate the metric that the equations determine. As remarked by Einstein in [1],

  "*[...] the requirement of general covariance takes away from space and time the last remnant of physical objectivity [...].*"

In the literature this property of the theory has frequently been described as *diffeomorphism-invariance*, together with the claim that general relativity is the only theory to possess such an attribute. Here we would like to argue in the direction of describing this property as a consequence of *general covariance*, or the *general principle of relativity*, together with the absence of an *a priori* structure. In fact any theory that respects this principle must be formulated as an action over a smooth manifold

$$S = \int_M \mathcal{L}(\phi) = \int_{\Phi(M)} \Phi^* \mathcal{L}(\phi)\,, \tag{1.9}$$

and the integral that describes it is invariant under a diffeomorphism $\Phi : M \to N$ of the manifold when the 4-form integrand is appropriately pulled-back. The equations of the theory are left completely unchanged, and this is not exclusive to general relativity. The special character of the theory of gravity lies rather in the absence of a background structure, which every other field theory has in the form of a Minkowski metric over an $\mathbb{R}^{3,1}$ manifold.

- *Relational observables*

  Finally, it follows from the discussion of the previous subsection that the observables of the theory must be *relational*, in the sense that they must be obtained through the description of some field in terms of another, rather than in terms of arbitrary coordinates without physical meaning. Again we recover the notion that the "spacetime" manifold lacks any true physical identity. As Rovelli humorously describes [8],

  "*No more fields on spacetime, just fields on fields. [...] we have to ride the whale.*"

1.2 Tetradic gravity

After a short review of the standard approach to general relativity, we now turn to a less known construction. There exist many alternative but equivalent formulations of the theory of gravity, all of them resting on different formal structures. Such an equivalence, we argue, is however not merely formal, for the choice of the pertinent mathematical objects one uses to describe the world carries with it an assignment of ontological importance to those objects. Moreover, in the realm of quantum physics, it is well-known that equivalent classical theories may give rise to inequivalent quantum ones. Considering a reformulation of the usual theory of spacetime, synthetically described by the Einstein-Hilbert action, is therefore most definitely not a wasted effort, and we will in fact argue for both the formal and conceptual usefulness of such a reformulation. In this section we will consider a first-order version of the Einstein-Hilbert lagrangian, and relax furthermore the standard assumption that one must use the Levi-Civita connection to describe gravity. The theory satisfying these two properties is usually called in the literature the *first-order Palatini formulation*, after the physicist [9] that first proposed considering the connection a *bona fide* physical field of the system, and it will turn out to be a gauge theory.

1.2.1 Frames of spacetime

To show how the usual description of the theory can be reconstructed into a first-order one, we start with the usual setting of General Relativity: a smooth 4-dimensional orientable Lorentzian manifold (M, g) and the unique Levi-Civita connection on the tangent bundle ∇_{LC} that one may use to transport vectors over the manifold. Now, the most standard way to describe the vectors of TM in a neighborhood of a point is through the canonical basis $\{\partial_\mu\}$ coming from a chart in that neighborhood, but this choice of basis, or frame of reference, may be too general for physical purposes. Indeed, in physics one is generally interested in considering *inertial frames*, because it is in these frames that physical laws seem to present themselves in the most simple way. Whether a frame is to be considered inertial or not depends of course on the metric that is associated to it, and we can take an inertial frame to be one where the metric looks like the Minkowski one.

Let us then consider at an open set $U \subset M$ an inertial frame, that is, a family of four local linearly-independent sections $\{e_I\}_{I=0,..,3}$, $e_I \in \Gamma(TU)$ satisfying an orthogonality condition with respect to the Minkowski metric η:

$$g(e_I, e_J) = \eta_{IJ} \,. \tag{1.10}$$

Such a choice of local sections can always be made [10], and we follow the common convention of calling the vector e_I either a *tetrad* or a *frame field*. Note that one can think of the frames as maps from the canonical basis $\hat{e}$ of $\mathbb{R}^{3,1}$,

$$\begin{aligned} e : M \times \mathbb{R}^{3,1} &\to TM \\ (x, \hat{e}_I) &\mapsto (x, {e_I}^\mu \partial_\mu) \,, \end{aligned} \tag{1.11}$$

and we denote $e_I = e(\hat{e}_I)$. Dual to the basis vectors of the frame there exist one-forms $\{\theta^I\}$, which we shall call *cotetrads*, or *coframe fields*, such that the metric can then be straightforwardly rewritten as

$$g = \eta_{IJ} \theta^I \otimes \theta^J \,, \tag{1.12}$$

and in this sense one may think of the coframe as a sort of square-root of the metric. The inner product of vectors with components $X = X^I e_I$, $Y = Y^J e_J$ relative to the frame is then simply $\langle X, Y \rangle = \eta_{IJ} X^I Y^J$, and as such we may think of the "internal" roman indices as Minkowski indices, and they can be raised and lowered with the Minkowski metric as usual. Moreover, given the canonical basis at the open set U, we may express the tetrads as $e_I = {e_I}^\mu \partial_\mu$ and the cotetrads as $\theta^I = {e^I}_\mu \, \mathrm{d}x^\mu$, such that from the orthogonality condition of equation (1.10) and the duality requirement $\theta^I(e_J) = \delta^I_J$, we find the equations

$$\begin{aligned} {e^I}_\mu {e_J}^\mu &= \delta^I_J \\ {e^I}_\mu {e_I}^\nu &= \delta^\mu_\nu \,, \end{aligned} \tag{1.13}$$

and the metric g on the manifold can be used to raise and lower "spacetime" greek indices. Note that the components ${e_I}^\mu$ have an important meaning: they are the functions that transform a generic frame into an inertial one, and this transformation is unique up to the Lorentz transformations $\Lambda \in SO(3,1)$ that relate different inertial frames, defined by

$$\eta_{IJ} = \Lambda^A{}_I \Lambda^B{}_J \eta_{AB} \,. \tag{1.14}$$

We can therefore always transform an inertial frame e_I to an equivalent one by the action of an $SO(3,1)$ matrix, and the resulting frame will still be orthogonal with respect to g.

What we have just described constitutes the conceptual gist of the tetrad framework, but there exists a very elegant way to mathematically systematize the objects we discussed. It turns out in fact [11] that the Lorentzian manifold (M,g) naturally gives rise to bundle of orthonormal frames ${}^{\perp}\mathrm{Fr}(TM)$, which may also be seen as a principal G-bundle $P(SO(3,1),M)$ (a bundle of coframes of T^*M can also be constructed in complete analogy). A local section of P will then be a local frame $e : M \times \mathbb{R}^{3,1} \to TM$, satisfying the orthogonality relation $g(e_I, e_J) = \eta_{IJ}$, and possibly being subject to a transformation through the right action of the Lorentz group $e_I \mapsto e_I \Lambda^I{}_J$. The appearance of a principal bundle suggests that we are in the presence of what can be seen as a gauge theory, albeit one that is unequivocally connected with the tangent bundle of M, in a way that a principal connection 1-form $\omega \in \Omega(P, \mathfrak{so}(3,1))$ of P naturally induces a connection on TM, and *vice-versa*. The gauge symmetry of the theory is precisely the one we already identified earlier with the transformation of a local frame by a Lorentz matrix. The interested reader in the mathematical description of how these structures arise is directed to Appendix A, where a brief review of frame and principal bundles, and their connections, is presented.

1.2.2 The first-order lagrangian

Taking into account the points of the previous subsection, the framework in which we presently want to position ourselves is the following: starting with a 4-dimensional orientable smooth manifold M, we consider a $G = SO(3,1)$ gauge theory formulated on the associated bundle $E \coloneqq P \times_\rho \mathbb{R}^{3,1}$ to the principal bundle $P(G,M) \xrightarrow{\pi} M$, with ρ as the fundamental representation of the Lorentz group.

Given a choice of trivializing maps $\phi_i : \pi^{-1}(U_i) \to U_i \times G$ over an open covering $\{U_i\}$ of M, we may define a set of canonical local sections $\sigma^i_I(x) = [\phi_i^{-1}(x,e), \hat{e}_I]$, where e is the identity element of G and $\hat{e}_I$ is the canonical basis in $\mathbb{R}^{3,1}$. A connection 1-form $\omega \in \Omega(P, \mathfrak{so}(3,1))$, with values in the Lie-algebra of the Lorentz group, can be pulled back to a form in some $U_i \subset M$ through $A^i = \phi_i^{-1}(x,e)^*\omega$, and in turn this matrix of forms specifies a connection in E (c.f. (A.20)) by acting on the sections as[1]

$$\nabla \sigma_I = \sigma_J \otimes \rho_*(A)^J{}_I \,. \tag{1.15}$$

The connection on E also determines a local curvature, written neatly with the exterior covariant derivative (A.21) as $F = \mathrm{D}A = \mathrm{d}A + A \wedge A$.

[1]We will omit both the symbol ρ and the set index i for convenience.

Now we intend to transport the above information to the tangent bundle of the base manifold. Since both E and TM are vector bundles of the same dimension, there exists an isomorphism

$$\begin{aligned} e : E &\to TM \\ \sigma_I &\mapsto e_I = e_I{}^\mu \partial_\mu \, , \end{aligned} \tag{1.16}$$

and we name *tetrads* the images e_I of the map. Both the differential operators ∇ and D can be made to act on TM through the isomorphism, so they act locally in an entirely analogous way as in E. Furthermore, we consider also the dual isomorphism acting on sections ς^I of the dual bundle E^*,

$$\begin{aligned} \theta : E^* &\to TM \\ \varsigma^I &\mapsto \theta^I = e^I{}_\mu \, \mathrm{d}x^\mu \, , \end{aligned} \tag{1.17}$$

and to the images θ^I we will call *cotetrads*. It is precisely in these isomorphisms that the degrees of freedom of the theory will lie: they determine in which manner the canonical sections of E induce frames in TM, and from this information a metric field may be constructed. Indeed, on each open set where the tetrads (more precisely, the σ_I) are defined[2], let the metric g be given by

$$g = \eta_{IJ} \theta^I \otimes \theta^J \, , \tag{1.18}$$

where η is the standard Minkowski metric in $\mathbb{R}^{3,1}$.

As an important step in making contact with the standard formulation of GR, we now show that the connection induced by ω turns out to be metric compatible. Consider two local sections $s = s^I e_I$, $t = t^J e_J$ of TM. We then have $\mathrm{d}g(s,t) = \mathrm{d}(s^I t^J)\eta_{IJ}$, and

$$\begin{aligned} g(\nabla s, t) + g(s, \nabla t) &= g(\mathrm{d}s^K e_K + e_K A^K{}_I s^I, t^J e_J) + g(s^I e_I, \mathrm{d}t^K e_K + e_K A^K{}_J t^J) \\ &= (\mathrm{d}s^K t^J + A^K{}_I s^I t^J)\eta_{KJ} + (s^I \, \mathrm{d}t^K + s^I A^K{}_J t^J)\eta_{IK} \\ &= \mathrm{d}s^I t_I + A_{JI} s^I t^J + s^I \, \mathrm{d}t_I + A_{IJ} s^I t^J \\ &= \mathrm{d}(s^I t^J)\eta_{IJ} \, , \end{aligned}$$

where we used $A_{IJ} = -A_{JI}$, as $\mathfrak{so}(3,1)$ can be identified with skew-symmetric matrices. We find that the condition for metric compatibility $\mathrm{d}g(s,t) = g(\nabla s, t) + g(s, \nabla t)$ holds.

So far, starting from a connection on an associated bundle and a choice of an isomorphism to the tangent bundle of the base, we have recovered a Lorentzian manifold (M, g) together with a metric compatible connection ∇. However, in order for the connection to be a Levi-Civita one, as it is usually considered to be, the connection must also have a vanishing torsion form. It turns out, as will be discusses below, that this condition can be made to arise naturally. To see how this is possible, we shall first rewrite the usual

[2]Of course, g is so-far only defined locally. However, we assume that a global metric exists, and operationally such a global metric is determined in much the same way one would in the standard Einstein-Hilber theory: by solving the equations for g at different open sets and then matching the solutions.

Einstein-Hilbert lagrangian (1.4) in terms of tetrads. Relating the curvature form F with the Ricci scalar through equation (A.34), and making use of the Hodge star as defined in (A.36), the curvature term can be recast in the form

$$\begin{aligned} \mathrm{d}^4x\sqrt{-\det g}\,R &= \delta^{\alpha\beta}_{[\mu\nu]}\cdot \mathrm{d}^4x\sqrt{-\det g}\,e_I{}^{\mu}e_J{}^{\nu}F^{IJ}{}_{\alpha\beta} \\ &= \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\epsilon^{\alpha\beta\rho\sigma}\,\mathrm{d}^4x\sqrt{-\det g}\,e_I{}^{\mu}e_J{}^{\nu}F^{IJ}{}_{\alpha\beta} \\ &= \mathrm{d}x^\alpha\wedge\mathrm{d}x^\beta\wedge\left(\frac{\sqrt{-\det g}}{2}\epsilon^{\mu\nu}{}_{\rho\sigma}\,\mathrm{d}x^\rho\wedge\mathrm{d}x^\sigma\right)e_{I\mu}e_{J\nu}F^{IJ}{}_{\alpha\beta} \\ &= F^{IJ}\wedge\star(\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu)e_{I\mu}e_{J\nu} \\ &= F^{IJ}\wedge\star(\theta_I\wedge\theta_J)\,, \end{aligned}$$

and the cosmological term becomes

$$\begin{aligned} \Lambda\sqrt{-\det g}\,\mathrm{d}^4x &= \Lambda\,\det\theta\,\frac{1}{4!}\epsilon_{\mu\nu\rho\sigma}\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu\wedge\mathrm{d}x^\rho\wedge\mathrm{d}x^\sigma \\ &= \frac{\Lambda}{4!}\epsilon_{IJKL}\delta^\alpha_{[\mu}\delta^\beta_\nu\delta^\gamma_\rho\delta^\delta_{\sigma]}e^I{}_\alpha e^J{}_\beta e^K{}_\gamma e^L{}_\delta\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu\wedge\mathrm{d}x^\rho\wedge\mathrm{d}x^\sigma \\ &= \frac{\Lambda}{4!}\,\epsilon_{IJKL}\,e^I{}_\alpha e^J{}_\beta e^K{}_\gamma e^L{}_\delta\,\mathrm{d}x^\alpha\wedge\mathrm{d}x^\beta\wedge\mathrm{d}x^\gamma\wedge\mathrm{d}x^\delta \\ &= \frac{\Lambda}{4!}\,\epsilon_{IJKL}\,\theta^I\wedge\theta^J\wedge\theta^K\wedge\theta^L \\ &= \frac{\Lambda}{12}\,\theta^I\wedge\theta^J\wedge\star(\theta_I\wedge\theta_J)\,. \end{aligned}$$

The reformulated action for gravity, which we will call the *tetrad action*, is then

$$S_T = \int_M F^{IJ}\wedge\star(\theta_I\wedge\theta_J) - \frac{\Lambda}{6}\,\theta^I\wedge\theta^J\wedge\star(\theta_I\wedge\theta_J)\,. \tag{1.19}$$

An important point to note is that, as it stands, the natural field to take as the object of the theory, with respect to which one varies the action, is the tetrad $\theta^I = e^I{}_\mu\,\mathrm{d}x^\mu$. The equations of motion for the system will then be written in terms of this field, and since the metric is, in a generalized sense, the square of the frames, the equations represent a sort of first-order reconstruction of the usual Einstein field equations. The physical content of the theory is therefore in how the tetrads in TM relate to the canonical sections of that bundle, and this relation directly represents the equivalence principle, one of the tenets of General Relativity, as we will later argue.

It remains to check whether the equations of motion that arise out of such an action are indeed the ones demanded by the theory of general relativity. This is the subject of the next subsection.

1.2.3 Equations of motion

As is well known, in General Relativity one is concerned only with the metric as the sole physical field of the theory. The task of transporting vectors along the manifold is relegated to the natural choice of the Levi-Civita connection as the unique connection on the tangent bundle that is both metric-compatible and torsion-free. Since the metric we have constructed is such that the connection induced by the gauge bundle is metric-compatible, one might ask whether it is possible to also make the vanishing of the torsion to somehow arise naturally from the theory. The answer, unsurprisingly, is that this can be done. Following Palatini, we demand of the connection A^{IJ} to be *itself a degree of freedom of the theory.*

The equations of motion can be obtained from the action principle. In what follows we use that the exterior covariant derivative acting on scalar-valued forms in $\Gamma(T^*M \otimes \mathbb{R})$ reduces to the exterior derivative. The variation with respect to the connection yields

$$
\begin{aligned}
\frac{\delta}{\delta A} &\to \int_M \delta F^{IJ} \wedge \star(\theta_I \wedge \theta_J) \\
&= \frac{1}{2}\epsilon_{IJAB} \int_M \mathrm{D}\delta A^{IJ} \wedge (\theta^A \wedge \theta^B) \\
&= \frac{1}{2} \int_M \mathrm{D}\left[\delta A^{IJ} \wedge \theta^A \wedge \theta^B\, \epsilon_{IJAB}\right] + 2\epsilon_{IJAB}\, \delta A^{IJ} \wedge \mathrm{D}\theta^A \wedge \theta^B \\
&= \frac{1}{2}\epsilon_{IJAB} \int_M \mathrm{d}\left[\delta A^{IJ} \wedge \theta^A \wedge \theta^B\right] + 2\delta A^{IJ} \wedge T^A \wedge \theta^B \\
&= \epsilon_{IJAB} \int_M \delta A^{IJ} \wedge T^A \wedge \theta^B
\end{aligned}
$$

$$
\Rightarrow\ T^A = 0\,, \tag{1.20}
$$

where we have used the definition of the torsion form from equation (A.35). This is exactly what we expected to find: the connection of the theory turns out to be the Levi-Civita connection, without it having to be an *ad-hoc* assumption. On the other hand, the variation with respect to the frames results in

$$
\begin{aligned}
\frac{\delta}{\delta\theta} &\to \int_M F^{IJ} \wedge \star(\delta\theta_I \wedge \theta_J \cdot 2) - \frac{\Lambda}{6}\,\delta\theta^I \wedge \theta^J \wedge \star(\theta_I \wedge \theta_J) \cdot 4 \\
&= \int_M 2\delta\theta_I \wedge \left[\theta_J \wedge \star F^{IJ} - \frac{\Lambda}{3}\,\theta_J \wedge \star(\theta^I \wedge \theta^J)\right]
\end{aligned}
$$

$$
\Rightarrow\ \epsilon_{IJAB}\left[F^{IJ} - \frac{\Lambda}{3}\,\theta^I \wedge \theta^J\right] = 0\,, \tag{1.21}
$$

which can easily be seen, as per equation (A.34), to simply be the Einstein field equations with a cosmological constant. We have therefore successful reformulated General Relativity, and its equations, in terms of a Lorentz gauge theory.

1.2.4 The meaning of the tetrads

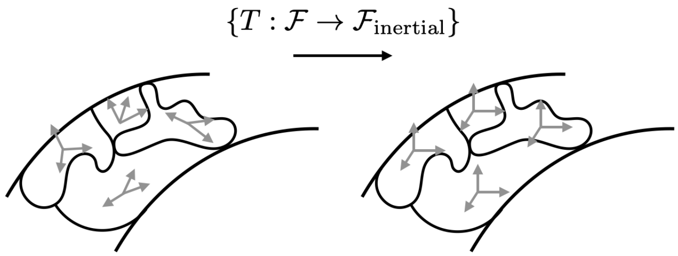


Figure 1.1: A spacetime with finite regions of matter generating gravitacional fields, and hence described by non-inertial frames $\mathcal{F}$. The set of transformations that take each frame to a non-inertial one can serve as a description of the gravitational field.

While the lagrangian in equation (1.19) seems different enough from the Einstein-Hilbert action, being constructed with different degrees of freedom, we may argue that it represents a more direct application of the physical principles described in subsection 1.1.1. Recalling the discussion there, the theory of general relativity is in essence a theory of reference frames. In the absence of matter, and more concretely in the absence of a gravitational field, one can imagine setting up a set of inertial frames over finite regions of spacetime (for example, by using clocks and rods such that Newton's law holds). Consider one of those frames over a region U, denoted by $\mathcal{F}_U$. As soon as matter is considered in this region, $\mathcal{F}_U$ becomes indistinguishable from a non-inertial frame. We can, however, return to it its inertial character by operating a transformation: we take $\mathcal{F}_U \to \mathcal{F}'_U$ such that $\mathcal{F}'_U$ is again an inertial frame. Note that this transformation is unique up to $SO(3,1)$ transformations; $\Lambda\mathcal{F}'_U$ is of course still inertial. In the general presence of matter, we can now describe the whole system by assigning an inertial frame to every region. However, we would like to describe the dynamics of matter with more precision; in fact we want to describe matter's position to the point, and this requires an assignment of a frame at every such point.

In this way, we may think of gravity as a theory which assigns an inertial frame at every point - these are our tetrad fields θ^I, defined earlier, which are indeed inertial because they satisfy the condition

$$g(\theta^I, \theta^J) = \eta^{IJ} \,. \tag{1.22}$$

The fact that given a non-inertial frame there is not a unique choice of a corresponding inertial frame is encoded in the gauge symmetry of the theory - the theory is essentially a gauge theory coupled to frame fields.

1.3 Other first-order formulations

We consider further possible actions for general relativity, in particular the Holst action and the constrained BF action. Both of these theories have proved very useful in several approaches to quantum gravity.

1.3.1 The Holst term

It is well-known that physical theories that are equivalent on-shell may give rise to inequivalent quantum versions of themselves, essentially because the path integral approach to quantum mechanics explicitly attributes a non-zero weight to trajectories of the system that do not extremize the action. In this subsection we consider an additional term one can add to the tetrad action, usually called the Holst term [12], which vanishes under the imposition of the equations of motion, corresponding therefore to a theory which is classically equivalent to Einstein's model, but not necessarily so at the quantum level. The resulting action, which we will call the Holst action, is given by

$$S_H = \int_M F^{IJ} \wedge \left(\star + \frac{1}{\gamma} \right) (\theta_I \wedge \theta_J) - \frac{\Lambda}{6} \theta^I \wedge \theta^J \wedge \star(\theta_I \wedge \theta_J) \,, \tag{1.23}$$

where $\gamma \in \mathbb{C} \setminus \{0\}$ is a yet to be fixed parameter, usually called the Immirzi parameter (this parameter corresponds to the one that appears in the canonical formulation of quantum gravity [13]). When γ is taken to be arbitrarily large, the frame action is recovered. Although this term is usually called a boundary term, it is important to note that it cannot be written as a total derivative, so its impotence really follows from the equations of motion. In a manner entirely similar to what was done for equation (1.20), we again find

$$T^I = 0 \,. \tag{1.24}$$

Substituting back in the lagrangian, the connection A^{IJ} is now completely determined by the vanishing of the torsion to be the Levi-Civita connection $A^{IJ}[\theta]$, so it is a fixed degree of freedom dependent on the tetrad. The Holst term then vanishes, since

$$\begin{aligned} F^{IJ}[\theta] \wedge \theta_I \wedge \theta_J &= F[\theta]^{IJ}{}_{\mu\nu} \theta_{I\alpha} \theta_{J\beta} \, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\alpha \wedge \mathrm{d}x^\beta \\ &= R[\theta]_{\mu\nu\alpha\beta} \epsilon^{\mu\nu\alpha\beta} \, \mathrm{d}^4 x \\ &= 0 \end{aligned}$$

where in the last line the Bianchi identity $R^\mu{}_{[\nu\alpha\beta]} = \nabla_{[\nu} T^\mu{}_{\alpha\beta]} + T^\mu{}_{\gamma[\nu} T^\gamma{}_{\alpha\beta]}$, which holds for any connection, was used. Still from this identity one understands the Achilles heel of the Holst action: it is only equivalent to the frame theory when the torsion vanishes, and when one considers matter fields the torsion field will not, in general, be zero. It is not only that the Holst theory may induce an inequivalent quantum theory to the Einstein-Hilbert one, but rather that in the presence of matter both theories are, even at the classical level, intrinsically different. Whether this circumstance is enough to deter one from using this action in an approach to quantum gravity is in the eye of the beholder. On the topic of the physical effects of a non-vanishing torsion many articles exist in the literature, and the interested reader is directed, for example, to Perez's enlightening work [14].

1.3.2 Constrained topological gravity

Yet another incarnation of general relativity, which has found considerable success in the descriptions of quantum gravity that will be discussed here, is the BF theory of gravity with constraints [15], so named because the theory is *background-free* in the sense that the action does not depend on a metric on the manifold (notice the absence of a Hodge star in (1.25)). The theory is formulated over a 4-dimensional spacetime M as the base manifold of the associated bundle $E = SO(3,1) \times_\rho \mathbb{R}^{3,1}$, just as before, with the isomorphism $e : E \to TM$. The relevant fields are a local connection 1-form A in M with values in $\mathfrak{so}(3,1)$, inducing a curvature form $F(A)$, and a 2-form B in M also with values in the algebra. Additionally, there is a matrix scalar field $\phi_{IJKL} = \phi_{KLIJ}$ that will serve as a Lagrange multiplier, which we define to be traceless, $\epsilon^{IJKL}\phi_{IJKL} = 0$. The action takes the form

$$S_{cBF} = \int_M B_{IJ} \wedge F^{IJ} - \frac{\Lambda}{12}\epsilon_{IJKL}B^{IJ} \wedge B^{KL} + \phi_{IJKL}B^{IJ} \wedge B^{KL}\,, \tag{1.25}$$

and Λ serves the purpose of a cosmological constant. The equations of motion for B and F directly yield

$$\frac{\delta}{\delta A} \to \mathrm{D}B^{IJ} = \mathrm{d}B^{IJ} + [A,B]^{IJ} = 0 \tag{1.26}$$

$$\frac{\delta}{\delta B} \to F_{IJ} = \frac{\Lambda}{12}\epsilon_{IJKL}B^{KL} - \phi_{IJKL}B^{KL}\,. \tag{1.27}$$

Varying the action with respect to the Lagrange multiplier one finds

$$\frac{\delta}{\delta \phi} \to B^{IJ} \wedge B^{KL} = \epsilon^{IJKL}\,V\,, \tag{1.28}$$

for some $V \in \Lambda^4 T^*M$. In the literature this set of equations is known as the *simplicity constraints*, since one can show that they constraint the form B^{IJ} to be simple, i.e. to be a wedge of one-forms (also called a bi-vector). That this is true was shown once and for all by Reisenberger in [16] using somewhat abstract geometrical arguments. Here, however, we present an easier algebraic proof of the result.

Proposition: The equation $B^{IJ} \wedge B^{KL} = \epsilon^{IJKL}\,V$ with non-vanishing V holds if and only if $B^{IJ} = \pm\theta^I \wedge \theta^J$ or $B^{IJ} = \pm\frac{1}{2}\epsilon^{IJ}{}_{KL}\theta^K \wedge \theta^L$.

Proof: It is easy to check that the equations for B^{IJ} satisfy the simplicity constraints. Regarding the other direction, it follows from the constraints that $B^{IJ} \wedge B^{KJ} = 0$. Now choose some local section $v = v^J e_J$ of TM such that $B^{IJ}(v) \neq 0$. Then $(B^{IJ} \wedge B^{KJ})(v) = 2B^{IJ}(v) \wedge B^{KJ} = 0 \Rightarrow B^{KJ} = \sum_I \alpha^{KI} \wedge 2B^{IJ}(v)$, for some matrix of 1-forms α^{KI}. This can be rewritten as $B^{IJ} = \alpha^I \wedge \beta^J$, with α^I and β^J two frames of TM.

Now we show colinearity. Since B takes values in $\mathfrak{so}(3,1)$, we have $B^{IJ} = -B^{JI}$. In terms of the one-forms,

$$\begin{aligned}
&\alpha^I \wedge \beta^J = -\alpha^J \wedge \beta^I \\
&\Leftrightarrow (\alpha^I{}_\mu \beta^J{}_\nu + \alpha^J{}_\mu \beta^I{}_\nu)\, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = 0 \\
&\Leftrightarrow \alpha^I{}_{[\mu} \beta^J{}_{\nu]} + \alpha^J{}_{[\mu} \beta^I{}_{\nu]} = 0 \\
&\Rightarrow \alpha^I{}_{[\mu} \beta^I{}_{\nu]} = 0\,,\ \forall I \\
&\Leftrightarrow \begin{vmatrix} \alpha^I{}_\mu & \alpha^I{}_\nu \\ \beta^I{}_\mu & \beta^I{}_\nu \end{vmatrix} = 0 \\
&\Leftrightarrow \exists k \in \mathbb{R} \quad \text{s.t.} \quad \alpha^I{}_\mu = k \beta^I{}_\mu\,,\ \forall \mu
\end{aligned}$$

Upon a suitable normalization we get $B^{IJ} = \pm\theta^I \wedge \theta^J$, as we intended. The other sector of solutions can also easily be found by noting that the simplicity constraint equation is invariant under $B^{IJ} \to \frac{1}{\sqrt{4!}}\epsilon^{IJ}{}_{KL}B^{KL}$, so the above argument can again be used. □

Constraining the B field to be simple, and defining a local metric over M as in (1.12), we find the actions

$$S'_{cBF} = \int_M F^{IJ} \wedge \theta_I \wedge \theta_J \pm \frac{\Lambda}{6}\,\theta^I \wedge \theta^J \wedge \star(\theta_I \wedge \theta_J) \tag{1.29}$$

$$S'^{\star}_{cBF} = \int_M F^{IJ} \wedge \star(\theta_I \wedge \theta_J) \pm \frac{\Lambda}{6}\,\theta^I \wedge \theta^J \wedge \star(\theta_I \wedge \theta_J)\,, \tag{1.30}$$

for $B^{IJ} = \pm\theta^I \wedge \theta^J$ and $B^{IJ} = \pm\star(\theta^I \wedge \theta^J)$, respectively. The action corresponding to the $\star$ sector turns out to be the tetrad action (1.19) up to a possible relative minus sign of the cosmological constant. Notice furthermore that if one chooses to sum the two actions with a relative $1/\gamma$ factor, corresponding to the two different sectors, the Holst action (1.23) is recovered.

Chapter 2

Towards a Quantum Theory of Gravity

The problem of theoretically unifying in a single physical model the theory of quantum mechanics and general relativity is evidently a difficult one, as one would conclude from the fact that it has remained unsolved for nearly a century now. While quantum theories of every other fundamental interaction have been successfully achieved in the framework of quantum field theory, all described in a common language and displaying common features, every attempt (up to now) to force the theory of gravity into that framework has been met with a certain inherent resistance from the theory itself. That internal resistance is most prominently represented by the seemingly unresolvable divergences that appear in perturbative metric computations around a specified background; the linearized Einstein-Hilbert quantum field theory is indeed famously non-renormalizable[1].

Does the difficulty in adapting general relativity to the extremely successful QFT framework hint at a idiosyncratic nature of gravity? We would argue it is so. Our discussion in Chapter 1 on the classical theory of general relativity intends to point out the many structural peculiarities of the world-view afforded by GR. Out of those, it is perhaps the characteristic of *absence of a-priori structures* in the theory, and the subsequent immateriality of a strong notion of time, that most conflicts with a perturbative QFT description, initially developed in the context of a Minkowski space-time. Going further still, if one thinks of gravity not as an interaction (as electromagnetism and the strong and week forces are) but rather as a consequential phenomenon of the geometrical properties of space-time, there is no reason to strongly argue that general relativity should admit a *naive* quantum field theory description as the other forces do; a different path would then be needed, implementing the lessons of subsection 1.1.4.

In this chapter we discuss in concrete physical terms a possible quantum model for gravity, with the chosen approach being the spin-foam one. The following is intended as a pedagogical review of both the general framework and a couple of specific models.

[1]The program of asymptotic safety, which has been showing considerable promise, aims precisely to find a renormalizable modification to the standard theory by studying its RG flow and demanding a finite coupling at asymptotic distances.

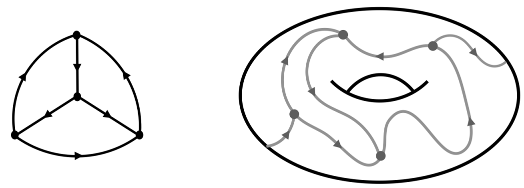


Figure 2.1: An oriented 3-valent graph and its embedding in a donut (also known as glazing a donut).

2.1 The quantum states of gravity

As we have just argued, a *naive* construction of the quantum states of gravity through the perturbative QFT prescription might not be the best option. To precede the exposition on spin-foam models, we now review a certain characterization of gauge theory states that seem adequate to the problem at hand: the so called *spin-network states.*

2.1.1 Gauge theory on a graph

We argued in the first chapter that the theory of general relativity can be recast as a gauge theory on the group of Lorentz transformations. In this section, following [17], we give the first steps towards a description of the Hilbert space $L^2(\mathcal{A}_\phi/\mathcal{G}_\phi)$ of a discretized gauge theory, given by the square-integrable, gauge-invariant functions on the space of connections over a graph.

Consider a principal G-bundle $P \xrightarrow{\pi} M$ of a compact Lie group. We can define an embedded graph in M using an equivalence relation on embedded curves; we define $\gamma_1 \sim \gamma_2$ if γ_2 can be obtained from γ_1 through an orientation-preserving homeomorphic reparametrization, and call the equivalence class $e = [\gamma]$ an *edge* in M. An *embedded graph* ϕ is then a collection of such edges such that they intersect only at their endpoints. We denote $\mathcal{E}_\phi$ and $\mathcal{V}_\phi$ the sets of edges and vertices of ϕ, respectively. To the vertex at the end of an oriented edge we call the target $t(e)$ of the edge, and analogously $s(e)$ for the source. Figure 2.1 is a pictorial representation of an embedded graph.

To construct a gauge theory over the graph ϕ, we restrict the principal bundle to $\mathcal{V}_\phi$, seen as the base space with the subset topology (which ends up being the discrete one). The resulting bundle is of course trivializable, and we have $P_\phi \simeq \mathcal{V}_\phi \times G$. By a well-known result from the theory of principal bundles, the connection on the bundle can be uniquely determined from all the parallel transports on the manifold. We adopt this view-point and consider the space of connections over the graph to be given by

$$\mathcal{A}_\phi = \prod_e \mathcal{A}_e \simeq G^{|\mathcal{E}_\phi|}\,, \tag{2.1}$$

where $\mathcal{A}_e$ is the space of edge bundle isomorphisms $A_e : P_{e(0)} \to P_{e(1)}$ that are G-compatible $A_e(xg) = A_e(x)g$, *i.e.* the space of parallel transports along the edge e. In turn, since gauge transformations act on parallel transports at their endpoints as in equation (A.17), we expect the gauge group $\mathcal{G}$ to act on the vertices. As discussed in subsection A.1.4, one has the identification $\mathcal{G}_\phi \simeq \prod_{v\in\mathcal{V}_\phi} P_v \times_{\text{conj}} G$ and an action $\mathcal{G}_\phi \triangleright \mathcal{A} : (gA)_e = g^{-1}_{e(1)} A_e g_{e(0)}$, so that in fact we may take

$$\mathcal{G}_\phi \simeq G^{|\mathcal{V}_\phi|}\,. \tag{2.2}$$

Now we want to describe the space $\mathcal{H} = L^2(\mathcal{A}_\phi/\mathcal{G}_\phi)$. Let Λ denote the set of all unitary irreducible representations of G, including the trivial one. Given the Hilbert space $L^2(G)$ at each edge, the gauge transformations

$$\begin{aligned} G \times G \triangleright G \\ (g_1, g_2) \mapsto g_2^{-1} g g_1 \end{aligned} \tag{2.3}$$

induce[2] a unitary representation of G on $L^2(G)$ through $U(g_1, g_2)f(g) = f(g_2^{-1} g g_1)$. We implement in this way the action of the gauge group in the space of functions of the connection. Using the Peter-Weyl theorem, discussed in subsection B.2.1, this space can then be decomposed into the sum of unitary representations of G with the isomorphism

$$L^2(\mathcal{A}_\phi) \simeq \bigotimes_{e\in\mathcal{E}_\phi} \bigoplus_{\lambda\in\Lambda} \mathcal{H}^\lambda \otimes \mathcal{H}^{*\lambda}\,, \tag{2.4}$$

and the gauge group acts on this space as

$$\bigotimes_{e\in\mathcal{E}_\phi} \bigoplus_{\lambda\in\Lambda} \rho^\lambda(g_{s(e)}) \otimes \rho^{*\lambda}(g_{t(e)})\,. \tag{2.5}$$

To proceed we may now use an interesting trick from Baez in [17]. Since for the product in the previous equation the associative property holds, we may equivalently write

$$L^2(\mathcal{A}_\phi) \simeq \bigoplus_{\Lambda\to\mathcal{E}_\phi} \bigotimes_{e\in\mathcal{E}_\phi} \mathcal{H}_e \otimes \mathcal{H}_e^*\,, \tag{2.6}$$

where the notation $\Lambda \to \mathcal{E}_\phi$ is meant to indicate a coloring of each edge of the graph ϕ by a unitary irreducible representation indexed by $\lambda \in \Lambda$. This decomposition can furthermore be factorized in terms of vertices: denote $\mathcal{S}_v \in \mathcal{E}_\phi$ the set of all edges whose source is v, and analogously the set $\mathcal{T}_v \in \mathcal{E}_\phi$ of edges whose target is v; then we may write the isomorphism

$$L^2(\mathcal{A}_\phi) \simeq \bigoplus_{\Lambda\to\mathcal{E}_\phi} \bigotimes_{v\in\mathcal{V}_\phi} \left(\bigotimes_{e\in\mathcal{S}_v} \mathcal{H}_e \bigotimes_{e\in\mathcal{T}_v} \mathcal{H}_e^* \right)\,. \tag{2.7}$$

With regards to the full Hilbert space $\mathcal{H} = L^2(\mathcal{A}_\phi/\mathcal{G}_\phi)$, notice that $L^2(\mathcal{A}_\phi/\mathcal{G}_\phi) \simeq \mathrm{Inv}\,(L^2(\mathcal{A}_\phi))$, where $\mathrm{Inv}\,(L^2(\mathcal{A}_\phi))$ denotes the invariant subspace under $\mathcal{G}_\phi$. In Section B.3 we show that we can characterize the space of invariants $\mathrm{Inv}_\theta(\bigotimes_{e\in\mathcal{S}_v} \mathcal{H}_e \bigotimes_{e\in\mathcal{T}_v} \mathcal{H}_e^*) \simeq$

[2] Unitarity follows from the fact that the Haar measure is bi-invariant on compact groups.

$\mathrm{Int}(\bigotimes_{e\in\mathcal{S}_v}\mathcal{H}_e, \bigotimes_{e\in\mathcal{T}_v}\mathcal{H}_e^*)$ in terms of intertwiners. Hence we may finally decompose the space of our theory as

$$L^2(\mathcal{A}_\phi/\mathcal{G}_\phi) \simeq \bigoplus_{\Lambda\to\mathcal{E}_\phi}\bigotimes_{v\in\mathcal{V}_\phi}\mathrm{Int}\left(\bigotimes_{e\in\mathcal{S}_v}\mathcal{H}_e, \bigotimes_{e\in\mathcal{T}_v}\mathcal{H}_e\right) \tag{2.8}$$

and a basis for this space can be written through a basis ι_v for the intertwiners as

$$L^2(\mathcal{A}_\phi/\mathcal{G}_\phi) \simeq \mathrm{span}\{\bigoplus_{\Lambda\to\mathcal{E}_\phi}\bigotimes_{v\in\mathcal{V}_\phi}\iota_v\}\,. \tag{2.9}$$

States in this space are called *spin-network states.* A general state will then have the form

$$|\psi\rangle = \bigoplus_{\Lambda\to\mathcal{E}_\phi}\bigotimes_{v\in\mathcal{V}_\phi}(c_v)^{i_1\ldots i_m}_{j_1\ldots j_n}(\iota_v)^{j_1\ldots j_n}_{i_1\ldots i_m} \tag{2.10}$$

where we use the matrix elements of the intertwiners and c_v encodes the components.

There is a natural way to construct explicitly a function of the connection from these states. Notice that the intertwiners are maps

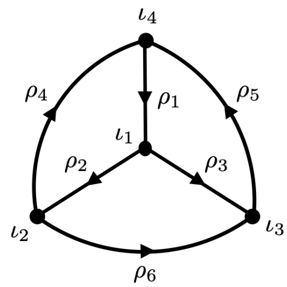
$$\iota_v : \mathcal{H}^v_{\mathrm{in}} \to \mathcal{H}^v_{\mathrm{out}}\,, \tag{2.11}$$

and that the same Hilbert space associated to an edge will be an incoming space as many times as it will be an outgoing one. To construct an element in $\mathbb{C}$, we may then simply compose all the intertwiners in the graph. The dependence of this number on a connection can be made, before the composition of the intertwiners, by acting on each intertwiner with a representation of some group element from the left and from the right. The group element should be determined by the connection, so we choose to assign it through the parallel transport of the connection along the edge. Formally, the *spin-network wave-function* for a basis state can be written as

$$\psi(A) = \left(\rho^{v_1}_{\mathrm{out}}(H_A^{e^{\mathrm{out}}_{v_1}}) \circ \iota^{v_1} \circ \rho^{v_1}_{\mathrm{in}}(H_A^{e^{\mathrm{in}}_{v_1}})\right) \circ \ldots \circ \left(\rho^{v_n}_{\mathrm{out}}(H_A^{e^{\mathrm{out}}_{v_n}}) \circ \iota^{v_n} \circ \rho^{v_n}_{\mathrm{in}}(H_A^{e^{\mathrm{in}}_{v_n}})\right)\,, \tag{2.12}$$

where $\rho^v_{\mathrm{in/out}}$ denotes the tensor product of incoming/outgoing representations at the vertex v, ι^v denotes the intertwiner at v and $H_A^{e_v^{\mathrm{in/out}}}$ stands for all the holonomies along the incoming/outgoing edges e at the vertex. They are given as always by $H^e_A = \mathcal{P}\exp\{-\int_e A\}$. In terms of indices, the wave function is found by assigning upper (for incoming representations) or lower (for outgoing) indices to the intertwiners, and then contracting them with the matrices associated with the holonomies.

As a closing remark, we would like to point out that spin-networks were essential in the development of loop quantum gravity [8]. Although they were only developed here in the context of a graph over a manifold, one of the substantial steps of loop quantum gravity was defining, in a rigorous manner, the space of gauge-invariant functions of the connection over the full manifold, rather than merely over an embedded graph. On this, see *e.g.* [18, 19, 20] and references therein.



$$\psi(A) = (\rho_1)^i_a(\iota_1)^a_{bc}(\rho_2)^b_j(\rho_3)^c_d(\rho_6)^k_e(\iota_3)^{de}_f(\rho_5)^f_h(\rho_4)^l_g(\iota_4)^{gh}_i(\iota_2)^j_{kl}$$

Figure 2.2: A basis spin network state labeled by a choice of representations on each edge and intertwiners at each vertex. The wave function associated to the state is found by contracting the indices according to the combinatorics of the diagram. We use the simplified notation $\rho_i = \rho_i(H^i_A)$.

2.1.2 Geometric observables

Having identified the Hilbert spaces we are interested in, observables in these spaces must now be introduced. In order to make a more concrete argument, we choose the specific Hilbert space

$$\mathcal{H} = \bigoplus_{j_0,j_1,j_2,j_3} \mathrm{Inv}_{SU(2)} \left(j_0 \otimes j_1 \otimes j_2 \otimes j_3 \right) , \tag{2.13}$$

corresponding, for $G = SU(2)$, with one of the tensor factors in equation (2.8) specified to four outgoing edges. The j_i denote unitary irreducible representations. In doing so we follow [21, 22].

Denoting by J^i the generators of $SU(2)$, satisfying the Lie bracket relation $[J^i, J^k] = i\epsilon^{ik}{}_l J^l$, the group representation with which $\mathcal{H}$ is constructed induces a representation of the algebra through the exponential map, well-known from quantum mechanics. We now define the operators[3]

$$\begin{aligned} B^i_0 &= J^i \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \\ B^i_1 &= \mathbb{1} \otimes J^i \otimes \mathbb{1} \otimes \mathbb{1} \\ B^i_2 &= \mathbb{1} \otimes \mathbb{1} \otimes J^i \otimes \mathbb{1} \\ B^i_3 &= \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes J^i \,. \end{aligned} \tag{2.14}$$

Since we are interested in quantum gravity observables we would like to construct operators associated to geometrical quantities, and to do so we must interpret the B^i_μ geometrically. Note that, because the B^i_μ generate the action of $SU(2)$ on $\mathcal{T} = \bigotimes_i j_i$, we have the

[3] We denote these operators by the letter B because these will precisely be the operators associated to the B field of the BF action later on. The fact that they represent faces should not be surprising, as in that theory the B field is a two dimensional object, which under the simplicity constraints is determined by the product of two edge vectors.

equivalent characterization of $\mathcal{H}$ as

$$\mathcal{H} \simeq \left\{ |\psi\rangle \in \mathcal{T} \;\middle|\; \sum_{\mu} B_{\mu} |\psi\rangle = 0 \right\} , \tag{2.15}$$

so we are led to think of the B^i_μ as imposing some kind of geometrical closure condition on the states. The most immediate choice of a geometrical object in three dimensions that satisfies such a constraint is the tetrahedron, with B_μ interpreted as the normal vector to the μth face. If this is the case, the area of each of these faces is classically just the norm of each vector, so we may define an *area operator* through the square of each B, that is

$$A_\mu = \sqrt{B_\mu \cdot B_\mu} , \tag{2.16}$$

but this is simply the Casimir L^2 of $SU(2)$. The area operator for each face μ is thus defined by the eigenvalues

$$A_\mu |\psi\rangle = \sqrt{j_\mu(j_\mu + 1)} |\psi\rangle , \tag{2.17}$$

from which one can also define a total area.

In complete analogy, we define moreover the *volume operator*. It is classically given by the triple product $V = \sqrt{\frac{1}{3!}|B_1 \cdot (B_2 \times B_3)|}$, so the operator can be written as

$$V = \sqrt{\frac{1}{3!}|\epsilon_{ijk} B^i_2 B^j_1 B^k_3|} , \tag{2.18}$$

and one can choose to define these operators with a multiplicative $\hbar$ constant.

An important remark to make is that there is a common interpretation of the area operator on spin-network states as giving the area of the surfaces intersected by the spin-network edges, comming from the loop quantum gravity construction [8]. This interpretation is consistent with the above definition, since each edge is associated with a representation. Later, when spin-foam models in 4 dimensions will be discussed (subsection 2.4.2), the reader can check that indeed each spin-network induced by the foam will have an edge intersecting a face of a simplex, so this interpretation is still reasonable in the spin-foam formulation.

2.2 Dynamis on a generic background

We have now constructed states and operators that could, in principle, be used in a quantum theory of gravity. Before putting them to good use, we must still discuss what approach we should take to a quantum theory on a generic spacetime that might not have the nice properties of the Minkowski universe.

In quantum field theory one is most frequently interested in a Minkowski background. Notions like asymptotic states and a global time with translational symmetry are then readily available, and one can construct scattering theory as usual. As remarked in the first chapter, we do not however expect a quantum theory of gravity to be formulated in such a stiff background. The absence of preferred background structures potentiates a myriad of different situations. In this light a reformulation of the basic ideas is in order. How are we to think about measurements, states, evolution, *etc.* on a generic background? In such a general situation, how should we interpret the spin-network states constructed above?

Robert Oeckl proposed such a formulation [23] in the framework he called the *general boundary formulation* (GBF), taking inspiration from Atiyah's topological field theories. The idea is simple. Operationally, one makes measures in quantum physics by establishing some *initial* data and measuring again the state of that data at some *final* instance, repeating this process many times and finding a probability frequency for each pair of initial and final data. Now, considering initial and final data means registering some properties of a system at two time hyper-surfaces. It is here that one defines the quantum mechanical state. Between those hyper-surfaces one has no capability of measurement, but one can use Feynman's sum-over-histories approach in the bulk to predict how the data at each hyper-surface is related. The most direct generalization consists then in conceiving of spacetime regions with boundary as submanifolds of the same dimension of the spacetime manifold, and assigning Hilbert spaces to boundaries of those regions. One then has to prescribe a way to associate a number to each state, and this should be done with a sum-over-histories over the region.

2.2.1 Postulates of GBF

To better understand the framework, we collect here the postulates of the general boundary formulation. Consider a 4-dimensional spacetime oriented manifold M. Once calls *regions* to 4-dimensional sub-manifolds $R \subset M$ with boundary Σ. Both the regions and the boundaries receive an induced orientation from M. If Σ has some orientation, we denote by $\bar{\Sigma}$ the same boundary with opposite orientation. Then we demand:

1. Associated to each boundary there is a Hilbert space $\mathcal{H}_\Sigma$. If the boundary is a disjoint union of boundaries $\Sigma = \bigcup_i \Sigma_i$, then the Hilbert space decomposes as $\mathcal{H}_\Sigma = \bigotimes_i \mathcal{H}_{\Sigma_i}$.

2. For each boundary Σ there is an antilinear involution $\iota_\Sigma : \mathcal{H}_\Sigma \to \mathcal{H}_{\bar{\Sigma}}$. It is compatible with 1. in the sense that if the Hilbert space decomposes then ι_Σ also decomposes as a tensor product.

3. For each boundary Σ there is a non-degenerate bilinear form $(\cdot,\cdot)_\Sigma : \mathcal{H}_\Sigma \otimes \mathcal{H}_{\bar{\Sigma}} \to \mathbb{C}$. The form is symmetric $(a,b)_\Sigma = (b,a)_{\bar{\Sigma}}$ and compatible with 1. in the sense that the form of a disjoint union of boundaries is the product of the form for each boundary. The form also induces the inner product in $\mathcal{H}_\Sigma$ through $\langle\cdot,\cdot\rangle_\Sigma = (\iota_\Sigma(\cdot),\cdot)_\Sigma$.

4. For each region R with boundary Σ there is a an *amplitude* map $\rho_R : \mathcal{H}_\Sigma \to \mathbb{C}$.

5. Suppose R is a region with boundary $\Sigma = \Sigma_1 \cup \Sigma_2$. Suppose that the amplitude map $\rho_R : \mathcal{H}_{\Sigma_1} \otimes \mathcal{H}_{\Sigma_2} \to \mathbb{C}$ induces an isomorphism $\tilde{\rho}_R : \mathcal{H}_{\Sigma_1} \to \mathcal{H}_{\Sigma_2}$. Then we demand $\tilde{\rho}_R$ to be unitary.

6. Let R_1, R_2 be two regions with boundaries $\Sigma_1 \cup \Sigma, \Sigma_2 \cup \Sigma$ such that $R_1 \cup R_2$ is again a region and the intersection is Σ. Then, if $\rho_{R_1}, \rho_{R_2}\rho_R$ induce maps $\tilde{\rho}_{R_1}, \tilde{\rho}_{R_2}, \tilde{\rho}_R$, then it must be that $\tilde{\rho}_R = \tilde{\rho}_{R_2} \circ \tilde{\rho}_{R_1}$.

The physical meaning of these postulates is clear. The first axiom follows from the preceding discussion. Postulate 2. identifies the boundary with opposite orientation to the original as its dual in the finite dimensional case. Axioms 3. and 4. construct an inner product on each Hilbert space and a way of assigning probabilities to states. Postulate 5. is needed to match quantum mechanics with conservation of probabilities, and 6. is a reasonable consistency condition on gluing.

2.2.2 The amplitude map

There are two important points that need clarification in this framework: the form of the amplitude map and the meaning of quantum probabilities. Regarding the first, remember from quantum mechanics that the Feynman propagator has the form

$$\langle x; 0 | x; t \rangle = \int_{y(0)=x(0)}^{y(t)=x(t)} \mathcal{D}y \; e^{\frac{i}{\hbar} S[y]} \,, \tag{2.19}$$

where one sums over every possible path $y(t)$ that agrees on the boundary with the positions $x(0), x(t)$, obtaining the transition amplitude between the eigenstate of the position operator $|x\rangle = |x; 0\rangle$ and the evolved state $|x; t\rangle = U(t, 0) |x\rangle$. The usual generalization one makes in conventional QFT on a Minkowski spacetime is obtained by considering the *field operator* Φ and its eigenstates $\Phi |\phi; \Sigma_{1,2}\rangle = \phi(\Sigma_{1,2}) |\phi; \Sigma_{1,2}\rangle$, where Σ_1, Σ_2 denote two time hypersurfaces at t_1, t_2 and $\phi(\Sigma_i)$ is the *field* evaluated at one of the surfaces. The generalized path integral becomes

$$\langle \phi; \Sigma_1 | \phi; \Sigma_2 \rangle = \int_{\xi(\Sigma_i)=\phi(\Sigma_i)} \mathcal{D}\xi \; e^{\frac{i}{\hbar} S[\xi]} \,, \tag{2.20}$$

where now the sum is over *field configurations* ξ that agree with the values of the field ϕ on the boundaries. Given two general states $|\bar{\psi}\rangle, |\bar{\varphi}\rangle$[4] defined on Σ_1, Σ_2 respectively, their transition amplitude can be computed as

$$\begin{aligned} \langle \bar{\psi} | \bar{\varphi} \rangle &= \int \mathrm{d}\alpha \, \mathrm{d}\beta \;\; \langle \bar{\psi} | \alpha; \Sigma_1 \rangle \langle \alpha; \Sigma_1 | \beta; \Sigma_2 \rangle \langle \beta; \Sigma_2 | \bar{\varphi} \rangle \\ &= \int \mathrm{d}\alpha \, \mathrm{d}\beta \; \bar{\psi}^*(\alpha, \Sigma_1) \bar{\varphi}(\beta, \Sigma_2) \int_{\substack{\xi(\Sigma_1)=\alpha(\Sigma_1) \\ \xi(\Sigma_2)=\beta(\Sigma_2)}} \mathcal{D}\xi \; e^{\frac{i}{\hbar} S[\xi]} \,, \end{aligned}$$

[4]Because we are using greek letters for both the field and the states, we put a bar over the greek letter $|\bar{\psi}\rangle$ to indicate that the state is a linear superposition of the eigenstates $|\phi\rangle$ of the field operator, in much the same way that one uses $|\psi\rangle$ to indicate a superposition of the position eigenstates $|x\rangle$.

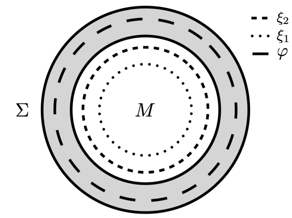


Figure 2.3: The amplitude associated to a state of a field configuration φ on the boundary is found by summing over every possible field configuration ξ in the bulk that agrees with the one on the boundary.

and the integrals in α, β are computed in the configuration space of the field ϕ at the respective boundaries. Note that $\bar{\psi}^*(\alpha, \Sigma_1)$ and $\bar{\varphi}(\beta, \Sigma_2)$ are wave-functionals of the configuration space at the boundaries. We may now use this special-case expression to propose an explicit form for the amplitude map ρ_M. Formally[5], we define it as

$$\begin{aligned} \rho_M : \mathcal{H}_\Sigma &\to \mathbb{C} \\ |\bar{\psi}\rangle &\mapsto \int \mathrm{d}\varphi \, \bar{\psi}(\varphi) \int_{\xi(\Sigma)=\varphi(\Sigma)} \mathcal{D}\xi \, e^{\frac{i}{\hbar}S[\xi]} \,, \end{aligned} \tag{2.21}$$

such that it reduces to the well-known case above in the appropriate situation. For a region with two disjoint boundaries and a unitary identification between the Hilbert spaces, we indeed have $\rho_M(|\varphi\rangle \otimes \langle\psi|) = \langle\psi|\phi\rangle$. Furthermore, this map incorporates the intuitive notion discussed above that the weight associated to a state should result from a sum-over-histories that are consistent with that state. In the special case where the state is a single field configuration $|\bar{\psi}\rangle = |\phi\rangle$, the amplitude reduces to the sum over all possible configurations that are consistent with the state $|\phi\rangle$.

2.2.3 Quantum probabilities

To close this section we turn to the interpretation of the probabilities assigned by the amplitude map. In conventional quantum mechanics, probabilities are usually motivated in the context of a *dynamical* interpretation of quantum theory. That it, some initial state *evolves* with some probability to a final state, and the probability one finds from the theory is exactly this one. Now, in much the same way that the framework discussed here attempts to generalize the usual Hilbert space construction to the situation where there does not exist a predefined background with nice properties, we must also upgrade our understanding of quantum probabilities in terms of a time evolution. A perhaps more useful way of thinking about these probabilities is rather in terms of *conditional* probabilities, as Oekl proposed. Note that we are associating states to boundaries without ever specifying their causal character, *i.e.* whether they are space- or time-like. This is indeed necessary

[5]It goes without saying that the rigorous mathematical formulation of the path integral will not be discussed here.

in a quantum gravity theory where a metric might not be readily at hand. As such, even if our spacetime region is composed of two disjoint boundaries, the amplitude associated to the tensor product of two states, which we usually think of as a transition amplitude associated to an evolution, could in principle be a correlation between states *at the same instant in time*. A truly general-relativistic approach to quantum probabilities must indeed allow for such correlations to exist, since we do not expect time itself to have any stronger ontological value than space. We then expect quantum probabilities not to be frequencies associated to *dynamics*, but rather to *space-time dynamics*, in the sense that any two states on the boundary of a region might be correlated by the laws of physics that describe the bulk between them.

Of course, our operational ability to measure these correlations might be limited, since we definitely might have a hard time making sense of what it means measuring a correlation of two states at the same instant in time. But we can make operationally meaningful statements by considering those subsets of conditional probabilities that we know how to measure. Quite generally, given a boundary, if we prepare a set of states $\mathcal{P}$ in a subregion of that boundary, and measure some set of states $\mathcal{M}$ in some other region, we would expect the correlation

$$\text{Prob}(\mathcal{M}|\mathcal{P}) = \frac{|\rho_M \circ P_{\mathcal{P}} \circ P_{\mathcal{M}}|^2}{|\rho_M \circ P_{\mathcal{P}}|^2}\,, \tag{2.22}$$

where $P_{\mathcal{P}}, P_{\mathcal{M}}$ are projectors into the respective subspaces.

We have now discussed both the construction of spin-network states and the meaning of quantum states in a general background. We are ready to attempt to formulate a theory of quantum gravity.

2.3 Spin-foams as discrete spacetime

2.3.1 Spin-networks from foams

Spin-network states for gravity were initially constructed in the context of a canonical approach to the quantum theory, today widely known as Loop Quantum Gravity (henceforth LQG). As Baez remarked in [21], the advent of spin-network states allowed for a "*rigorous and compelling picture of the* kinematic *aspects [...] of quantum gravity*". However, certain technical necessities in the development of the model, chief among them the construction of the states only at each spacial hyper-surface of a globally hyperbolic manifold, hindered a good understanding of the *dynamical* aspects of the theory. The spin-foam construction, which will be reviewed here and in the following sections, was developed to address this concern.

As we argued in Section 2.2, the dynamics of a theory in a general spacetime is to be understood as an assignment of a probability amplitude to boundary states, such that a *transition* amplitude is recovered in the special case where the boundary consists of two disjoint regions. In this way we want to formulate a model where spin-networks are assigned to boundaries of spacetime regions. The most intuitive way of associating spin-networks to 3-dimensional sub-manifolds of spacetime (the boundaries of the regions) is to consider a higher-dimensional analogue of a spin-network inside the spacetime manifold. That is, since a spin-network is in essence a colored combinatorial object with vertices and edges, we want to think of a colored higher-dimensional construction with vertices, edges and faces, in such a way that sections of that object become spin-networks. These higher-dimensional objects are commonly called spin-foams, from the foamy picture they evoke.

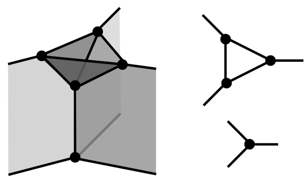


Figure 2.4: A spin-foam is a higher dimensional analogue of spin-network, made up of colored faces, edges and vertices. In this example one can identify a one-vertex spin-network in the lower part of the foam and a three-vertex one on the top.

There are many ways to specify rigorously what such a higher dimensional object might be. One such way is using *2-dimensional piecewise linear cell-complexes* (which we will refer to as simply *complexes* from now on), as is done in [21]. We will not need the concrete definition of these in our discussion beyond the loose concept of "flat" oriented vertices, edges and faces in $\mathbb{R}^n$, so the reader interested in the mathematical definition is directed to the Appendix of [21]. Now we can define a spin-foam:

Definition 2.3.1. A spin-foam F is a triple (κ, ρ, ι) consisting of:

1. A 2-dimensional oriented complex κ.
2. A labeling ρ of each face f of κ by an irreducible representation ρ_f of G.
3. A labeling ι of each edge e of κ by an intertwiner (see Appendix B for the definition of these objects)

$$\iota_e : \bigotimes_{f \in \mathcal{S}(e)} \rho_f \to \bigotimes_{f' \in \mathcal{T}(e)} \rho_{f'} \,, \tag{2.23}$$

where, in analogy to the notation used for the spin-networks, $\mathcal{S}(e)$ denotes the set of faces whose *source* is the edge e, that is, those faces meeting at e whose orientation agrees with the edge one. The set of faces meeting at an edge with opposite orientation, *i.e* those whose *target* is e, is denoted $\mathcal{T}(e)$.

To prescribe how a spin-network is induced by a spin-foam at its boundary we use the following definition, inspired by [24]:

Definition 2.3.2. The boundary $\partial\kappa$ of the complex κ is a subcomplex $\partial\kappa \subset \kappa$ such that there exists an injective, orientation-preserving affine map $c : \partial\kappa \times [0,1] \to \kappa$ mapping $\partial\kappa \times [0,1)$ to an open subset of κ. The interior is defined as $\mathring{\kappa} = \kappa \setminus \partial\kappa$. The boundary spin-network ∂F induced at the boundary of $F = (\kappa, \rho, \iota)$ is the colored oriented graph $\partial F = (\partial\kappa, \partial\rho, \partial\iota)$ obtained by assigning to each boundary edge $\bar{e} \in \partial\kappa$ the element $\partial\rho_f$ and to each boundary vertex $\bar{v} \in \partial\kappa$ the element $\partial\iota_e$, defined as[6]

$$\bar{e}: \ \partial\rho_f = \begin{cases} \rho_f & \text{if } f \in \mathcal{S}_{\bar{e}} \\ \rho_f^* & \text{if } f \in \mathcal{T}_{\bar{e}} \end{cases} \qquad \bar{v}: \ \partial\iota_e = \begin{cases} \iota_e & \text{if } e \in \mathcal{S}_{\bar{v}} \cap \mathring{\kappa} \\ \iota_e^* & \text{if } e \in \mathcal{T}_{\bar{v}} \cap \mathring{\kappa} \end{cases} \quad . \tag{2.24}$$

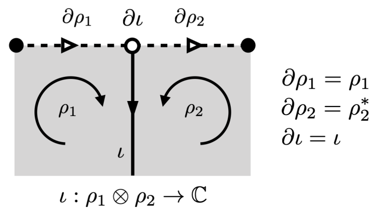


Figure 2.5: An example of how to label the boundary spin-network from a spin-foam. The faces of the spin-foam are colored, and the boundary is drawn with a dashed line.

2.3.2 Spacetime as a sum-over-foams

Now that we have the structure of a spin-foam in place, we come to a very important question: how exactly should we expect a spin-foam to describe something like a quantum spacetime?

Classical spacetime is modeled as a smooth Lorentzian manifold. Our experience with quantum theories, however, leads us to have two main expectations for a possible model of quantum spacetime:

- The dynamics of the objects should arise out of a sum over weighted possibilites; this is the Feynman sum-over-histories interpretation.
- The continuity of classical structures should be recovered from a limiting or averaging procedure of objects that are actually discrete.

The proposal of spin-foams is then the follwoing: interpret the classical spacetime manifold as the classical limit (meaning the most probable configuration) of a sum over spacetimes, and abstract the smoothness of the classical manifold as an approximation of piece-wise flat structures, obtained by some continuum limit. Each term of the sum, that is each possible spacetime state, would be described by a spin-foam.

In this context, the most honest way of constructing a quantum theory would probably be to fix a boundary state, *i.e.* a spin-network, consider every possible spin-foam that is coherent with that boundary state, assign a weight to each foam and sum over every

[6]Note that, unlike what is done in the literature, we do not consider "open" spin-foams, that is spin-foams where the boundary is not labeled. The reason for this will become clear in the discussion of the vertex spin-network.

structure to compute the amplitude of the state. However, much like every other area of physics, our ability to construct quantum theories is heavily constrained by our classical experience of the world. It is for this reason that we have a *quantization procedure* for classical theories rather than starting from scratch. A less ambitious but perhaps more fruitful way of obtaining a quantum theory of spacetime would on the other hand be extracting spin-foams from an already existing smooth manifold. This is moreover necessary for the use of spin-network states because, although they are in some sense fundamentally combinatorial and algebraic objects, our interpretation of them as being associated to functions of a smooth connection still strongly depends on their embedding on a manifold. Going forward we will therefore adopt the prespective of spin-foams as arising out of a background manifold through the concept of *complex dual to a triangulation*.

2.3.3 Simplicial spin-foams

A *triangulation* of a topological space X by a simplicial complex κ is a homeomorphism $f : X \to \kappa$, where we think of κ as the *geometrical realization* (as a topological space) of the combinatorial simplicial complex [25]. It turns out that simplicial complexes admit a corresponding *dual 2-complex* (sometimes called the *Poincaré dual*, or the *dual polyhedron*) [26], obtained by associating to the barycenter of every n-simplex a vertex, to every $(n-1)$-simplex an edge and to every $(n-2)$-simplex a face. In this way, every triangulation of a manifold associates to it a 2-complex, and we can use this as the complex of a spin-foam. Let us consider then a n-dimensional spacetime manifold and a triangulation Δ inducing

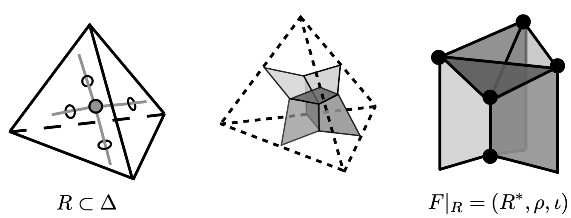


Figure 2.6: The minimal region of a triangulation by 3-simplices is the tetrahedron R. Associated to it there is a 2-complex R^* and a spin-foam $F|_R$.

the dual complex Δ^*. The triangulation decomposes the spacetime manifold into minimal pieces, the n-simplices R which we may think of as the regions of GBF discussed in Section 2.2. Moreover, since every n-simplex is bounded by a set Σ of $(n-1)$-simplices, each R has a natural boundary, so the dual complex Δ^* associates to each spacetime region a vertex, and to each boundary component an edge. Now, following the rationale of GBF, we would like to use a spin-foam $F = (\Delta^*, \rho, \iota)$ to induce states on each region. Realizing that we can think of the dual to each region R^* as 2-complex, the spin-foam F induces spin-foams in each region $F_R = (R^*, \rho, \iota)$ simply by restriction $F_R = F|_R$, and each of these spin-foams has a natural spin-network ∂F_R associated to it through definition (2.3.2).

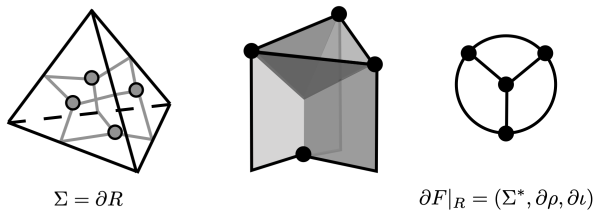


Figure 2.7: The boundary Σ of each minimal region is associated to the boundary spin-network ∂F of a minimal spin-foam.

We see in this way that a triangulation is a method for prescribing both *the set of Hilbert spaces of the theory* and the *combinatorial structure* used for assigning amplitudes. States on each boundary Σ are described by the coloring of $\partial F_R = (\Sigma^*, \partial\rho, \partial\iota)$, and amplitudes of those states are found by colorings of F_R consistent with the boundary. One can then choose which region to consider, perhaps as a union of each minimal region, and then focus on the associated boundary. In full correspondence with the amplitude maps of Section 2.2, we expect each boundary state to have the amplitude

$$\rho_R(\partial F_R) = \sum_{\substack{\{\iota\}\to\{e\} \\ \{\rho\}\to\{f\}}\big|_{\partial F_R}} A(F_R)\,, \tag{2.25}$$

where A is a weight associated to each foam, eventually derived from a lagrangian theory, and the notation under the sum indicates that the coloring of R^* must be consistent with the coloring of Σ^*.

As an immediate consequence, this construction allows us to assign to each vertex in Δ^* a probability amplitude computed from the spin-network state associated to it, and hence each simplex of the triangulation will have a corresponding weight. This weight is dependent on data on the faces of the spin-foam, which we can interpret as data on the $(n-2)$-simplices bounding each n-simplex. If we restrict to the particular case of a triangulation of a 4-dimensional manifold we thus recover the *quantum tetrahedron* of [27], and we can think of the data on the faces of each tetrahedron as assigning an *area state* to each face[7].

2.4 Spin-foam models

We will now discuss concrete spin-foam theories based on this framework. To see how the structure we have been constructing can be used in a theory of gravity, we will discuss first the simple case of a possible quantization of general relativity in three dimensions. Later we review the well-known EPRL model, a *bona-fide* quantum gravity theory.

[7]Of course, one needs closure conditions to make sure that the set of 4 faces really makes a tetrahedron.

2.4.1 Riemannian 3d gravity: BF theory

General relativity in three dimensions happens to have a particularly simple form. As we did in subsection 1.2.2 for the four-dimensional case, one can show that the action for general relativity in $(2+1)$ dimensions without cosmological constant is simply

$$S_T^{(3)} = \int_M \epsilon_{IJK} F^{IJ}[A] \wedge \theta^K \,, \tag{2.26}$$

where, as before, F^{IJ} is the curvature 2-form on TM induced from the local connection A on a principal G-bundle $P(G, M)$, and $\theta^I = \theta^I{}_\mu \, \mathrm{d}x^\mu$ is the image of an isomorphism $E^* \to T^*M$ mapping from the associated vector bundle $E^* = P \times_\rho \mathbb{R}^{2,1}$, which we will call the *triad*. If we take the case $G = SO(3)$, which is the correct group for three dimensional Riemannian gravity, we can identify this action with the well-known topological BF theory

$$S_{BF} = \int_M \mathrm{Tr}(F[A] \wedge B) \,, \tag{2.27}$$

where now B is an $\mathfrak{so}(2,1)$-valued 1-form. This is because there is an isomorphism $T^*M \to \mathfrak{so}(3)$ [28]. As was to be expected, the equations of motion of this theory look like a simpler form of (1.26):

$$\frac{\delta}{\delta A} \to \mathrm{D}B^{IJ} = \mathrm{d}B^{IJ} + [A, B]^{IJ} = 0 \tag{2.28}$$

$$\frac{\delta}{\delta B} \to F_{IJ} = 0 \,. \tag{2.29}$$

The theory restricts the curvature to be flat and the exterior covariant derivative of B to vanish. Note that, besides diffeomorphism invariance[8], the theory has two types of gauge symmetry

$$\delta A = \mathrm{D}\eta \quad \delta B = [B, \eta] \tag{2.30}$$

and

$$\delta A = 0 \quad \delta B = \mathrm{D}\eta \,, \tag{2.31}$$

where η is an $\mathfrak{so}(3)$-valued function. Since all flat connections are equal up to gauge transformations, and $\mathrm{D}B = 0$ implies by Poincaré's lemma that locally $B = \mathrm{D}\alpha$ for some form α [29], we see from the gauge symmetries that this theory has no degrees of freedom; it is completely constrained. Furthermore, by choosing a particular topology $M = \mathbb{R} \times M'$ (locally, every manifold has this form) and the temporal gauge $A_0 = 0$, one can easily see that $\frac{\partial \mathcal{L}}{\partial \dot{A}} = B$, implying that B is the momentum conjugate of A. We are therefore interested in the space $L^2(\mathcal{A}/\mathcal{G})$, so we can apply our spin-foam machinery.

[8]As discussed, all field theories are diffeo-invariant. However, usually in field theory one considers transformations by diffeos of all fields except the metric, finding, as we expect, that the theory is usually not invariant under such transformation. In BF theory there is no metric, so the "diffeomorphism" invariance is automatically satisfied.

Let us now introduce an oriented triangulation Δ of M, and consider its oriented dual 2-complex Δ^*. The formal path integral for the theory is

$$\begin{aligned} Z(M) &= \int \mathcal{D}A\mathcal{D}B \; e^{i\int_M \mathrm{Tr}(F[A]\wedge B)} \\ &= \int \mathcal{D}A \; \delta(F[A])\,, \end{aligned}$$

where we formally integrated over B to obtain the Dirac delta. In much the same way that we did in the context of spin-networks in subsection 2.1.1, we can discretize this theory on the dual complex Δ^*: we associate parallel transports of A to the edges $e \in \mathcal{E} \subset \Delta^*$, inducing in each face $f \in \mathcal{F} \subset \Delta^*$ a curvature as a product of group elements, *i.e.* an holonomy, which is known to relate to the curvature via the Ambrose-Singer theorem. The discrete path integral then takes the form

$$Z(\Delta^*) = \int \prod_{e\in\mathcal{E}} \mathrm{d}g_e \prod_{f\in\mathcal{F}} \delta\left(\prod_{e\in\partial f} g_e\right)\,, \tag{2.32}$$

where we are using the bi-invariant Haar measure. Using the tools from harmonic analysis in locally compact groups described in Section B.2, one can easily show that the delta function can be decomposed as

$$\delta(g) = \sum_{\lambda\in\Lambda} \dim(\rho_\lambda)\chi_{\rho_\lambda}(g)\,, \tag{2.33}$$

where the sum is taken over the irreducible unitary representations of $SO(3)$. The partition function then expands as

$$\begin{aligned} Z(\Delta^*) &= \int \prod_{e\in\mathcal{E}} \mathrm{d}g_e \prod_{f\in\mathcal{F}} \left(\sum_{\lambda\in\Lambda} \dim(\rho_\lambda)\mathrm{Tr}\left[\prod_{e\in\partial f} \rho_\lambda\left(g_e\right)\right]\right) \\ &= \sum_{\Lambda\to\mathcal{F}} \int \prod_{e\in\mathcal{E}} \mathrm{d}g_e \prod_{f\in\mathcal{F}} \left(\dim(\rho_f)\mathrm{Tr}\left[\prod_{e\in\partial f} \rho_f\left(g_e\right)\right]\right) \\ &= \sum_{\Lambda\to\mathcal{F}} \left[\prod_{f\in\mathcal{F}} \dim(\rho_f)\right] \int \prod_{e\in\mathcal{E}} \mathrm{d}g_e \; \mathrm{Tr}_{f\in\mathcal{F}}\left[\prod_{f\in\mathcal{F}} \left(\prod_{e\in\partial f} \rho_f\left(g_e\right)\right)\right] \\ &= \sum_{\Lambda\to\mathcal{F}} \left[\prod_{f\in\mathcal{F}} \dim(\rho_f)\right] \mathrm{Tr}_{f\in\mathcal{F}}\left[\prod_{e\in\mathcal{E}} \left(\int \mathrm{d}g_e \prod_{f:\, e\in\partial f} \rho_f\left(g_e\right)\right)\right] \end{aligned}$$

where in the first equality we again used the trick of subsection 2.1.1 to interchange the product over faces with the sum over labels, and in the last line $\mathrm{Tr}_{f\in\mathcal{F}}$ denotes contraction of the indices that follow *for each single face.* Referring to Section B.3, we identify the

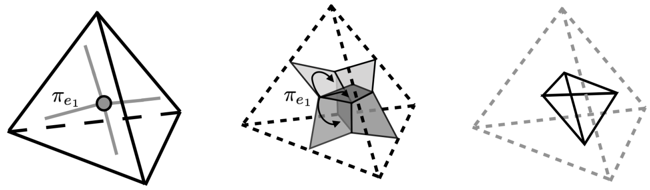


Figure 2.8: The path integral assigns to each dual edge a projector from the tensor product of three representations, one for each dual face. After connecting the arrows that lie on the same face one gets a 6j symbol for each tetrahedron.

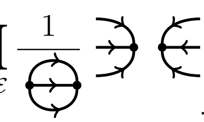
argument of the trace to be a product for each edge of projector maps

$$\begin{aligned} \pi_e : \bigotimes_{f\in\mathcal{S}(e)} \mathcal{H}_{\rho_f} \bigotimes_{f\in\mathcal{T}(e)} \mathcal{H}^*_{\rho_f} &\to \mathrm{Inv}\left(\bigotimes_{f\in\mathcal{S}(e)} \mathcal{H}_{\rho_f} \bigotimes_{f\in\mathcal{T}(e)} \mathcal{H}^*_{\rho_f}\right) \\ \pi_e = \int_{SU(2)} \mathrm{d}g_e \bigotimes_{f\in\mathcal{S}(e)} \rho_f(g_e) &\bigotimes_{f\in\mathcal{T}(e)} \rho^*_f(g_e)\,, \end{aligned} \tag{2.34}$$

which we may also think of as intertwiners $\iota_e : \bigotimes_{f\in\mathcal{S}(e)} \mathcal{H}_{\rho_f} \to \bigotimes_{f\in\mathcal{T}(e)} \mathcal{H}^*_{\rho_f}$. For three $SO(3)$ representations there is only one scalar invariant subspace, and using the diagrammatic notation of Appendix C we may then write

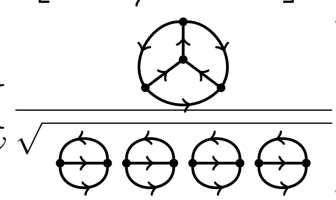
$$Z(\Delta^*) = \sum_{\Lambda\to\mathcal{F}} \left[\prod_{f\in\mathcal{F}} \dim(\rho_f)\right] \mathrm{Tr}_{f\in\mathcal{F}} \left[\prod_{e\in\mathcal{E}} \pi_e\right]$$

$$= \sum_{\Lambda\to\mathcal{F}} \left[\prod_{f\in\mathcal{F}} \dim(\rho_f)\right] \mathrm{Tr}_{f\in\mathcal{F}} \left[\prod_{e\in\mathcal{E}} \right.$$



$$\left.\right]$$

$$= \sum_{\Lambda\to\mathcal{F}} \left[\prod_{f\in\mathcal{F}} \dim(\rho_f)\right] \left[\prod_{v\in\mathcal{V}} \right.$$



$$\left.\right], \tag{2.35}$$

where in the last equation we used the fact that there are four edges for each vertex $v \in \mathcal{V} \subset \Delta^*$, and two vertices for each edge. The way the projectors are contracted is dictated by the face traces. We have omitted labels, but they can be reinstated by referring to equation C.26. We have thus found the partition function of BF theory in the spin-foam formulation to be a sum over colorings, with the weight of each coloring being an assignment of a 6-j symbol to each vertex of the dual complex, *i.e.* to each tetrahedron of the triangulation. Notice that the 6-j symbol has the exact same combinatorial structure as the boundary spin-network of a tetrahedron spin-foam as in Figure 2.7, so we may think of each symbol as the amplitude corresponding to each boundary spin-network. Referring back to equation (2.25), this is exactly what we expect from a spin-foam amplitude ρ_R, with the spin-foam weight

$$A(\partial F|_R) = \left[\prod_{f\in(\mathcal{F}\cap R^*)} \dim(\rho_f) \right] \left[\prod_{v\in(\mathcal{V}\cap R^*)} 6j \right] . \tag{2.36}$$

Closing our eyes to possible regularization problems of the above expression, we have succeeded at constructing the BF spin-foam theory.

One last comment must be made regarding a small nuance on the combinatorics of the $6j$-symbol appearing in equation (2.35). Recall that the edge orientations in the diagrams encode the domain-codomain structure of the invariant element, according to what was defined in Appendix C. The orientations on the clebsches in the second line of (2.35) depend therefore on the agreement between face and edge orientations of the dual complex, as in equation (2.5). In order to take the trace over the matrix elements, *i.e.* contract the $3j$-symbols into a $6j$-symbol, one needs to use either the symbol itself or its hermitian conjugate (diagrammatically, the left or the right symbol on the second line of (2.35)). One can check [24] that a coherent choice for the contraction is given by

$$\mathrm{Tr}\left[\bigotimes_{\mathcal{S}(v)} \iota^\dagger \bigotimes_{\mathcal{T}(v)} \iota \right] = 6j\,. \tag{2.37}$$

Note that, due to this requirement, changing the orientation of an edge *does not change the orientations of the diagrams*, because the conjugation generated by a different orientation relative to the face is compensated by the conjugation necessary to take the trace. The orientations in the diagrams therefore change only if a change in a face orientation is operated (furthermore ensuring coherence of the diagram).

2.4.2 Lorentzian 4d gravity: the EPRL model

Now we tackle a full-fledged model of quantum gravity. In subsection 1.3.2 we showed that general relativity can be written as a constrained BF theory. Since we already have a functional spin-foam model for BF theory, this suggests that we attempt to construct the quantum gravity theory starting from it. We will therefore consider the action of equation (1.25) without cosmological constant for $G = SL(2,\mathbb{C})$ (the double cover of the Lorentz group),

$$S_{cBF} = \int_M B_{IJ} \wedge F^{IJ} + \phi_{IJKL} B^{IJ} \wedge B^{KL}\,, \tag{2.38}$$

which is classically equivalent to the topological BF theory with an additional simplicity constraint

$$S_{BF} = \int_M B_{IJ} \wedge F^{IJ}\,, \qquad B^{IJ} \wedge B^{KL} = \epsilon^{IJKL}\, V\,, \tag{2.39}$$

where B^{IJ}, F^{IJ} are $\mathfrak{sl}(2,\mathbb{C})$-valued 2-forms and V is some non-vanishing 4-form. Since the simplicity constraint forces the B field to be a wedge of tetrads $B^{IJ} = \pm\theta^I \wedge \theta^J$ or $B^{IJ} = \pm \star (\theta^I \wedge \theta^J)$, we may recover the Holst action (1.23) by writing[9]

$$S = \int_M F^{IJ} \wedge \left(1 + \frac{1}{\gamma}\star\right) B_{IJ} \,, \qquad B^{IJ} = \pm \star (\theta^I \wedge \theta^J) \tag{2.40}$$

A possible way forward is hence to quantize the above BF version of the Holst theory, as we have done in the section before, and apply the simplicity constraints at the quantum level. This implies modifying the previous spin-foam model in three ways: we have to go one dimension higher, we have to consider the Lorentzian case and we have to constrain the states. Going to four dimensions is in principle a trivial matter; one needs only to consider in equation (2.34) projectors with four legs. The other two points require more discussion.

The Lorentzian case

We focus on the right regular representation of $SL(2,\mathbb{C})$ on the space of square integrable functions on that group. As stated in Appendix D, the unitary representations ρ^χ of $SL(2,\mathbb{C})$ are constructed on the Hilbert spaces H_χ of $SU(2)$ functions satisfying a covariance condition. Each Hilbert space is labeled by two complex numbers $\chi = (n_1, n_2)$ such that their difference is an integer. In the particular cases when $n_1 = -n_2^*$ one may relabel the spaces by an integer n and a real number p, denoting now $\chi = (n,p)$, and this is called the principal series of unitary representations. A basis for the principal series is given by the set

$$\left\{ \varphi^\chi_{j,m} = \sqrt{2j+1}\, D^j_{\frac{n}{2}m} \;\middle|\; j \geq \left|\frac{n}{2}\right|, |m| \leq j \right\} , \tag{2.41}$$

where D^j are the $SU(2)$ Wigner matrices, and in this $|\chi; j, m\rangle$ basis we may construct analogous $SL(2,\mathbb{C})$ Wigner matrices $D^\chi_{j_1 m_1 j_2 m_2}(g) = \langle \chi; j_1, m_1 |\, \rho^\chi(g)\, |\chi; j_2, m_2\rangle$ as in equation (D.10). A Fourier transformation can be constructed for $L^2(SL(2,\mathbb{C}))$ functions using this basis, and there exists a Plancherel theorem

$$f(g) = \frac{1}{2} \sum_{j,m,l,q} \int \mathrm{d}\chi \; (n^2 + p^2) \int_{SL(2,\mathbb{C})} \mathrm{d}h \; (D^*)^\chi_{jmlq}(g) f(h) D^\chi_{jmlq}(h) \,, \tag{2.42}$$

where $\int \mathrm{d}\chi = \sum_n \int \mathrm{d}p$, allowing for the L^2 space to be decomposed into a direct integral of the principal series H_χ spaces

$$\int_\oplus \mathrm{d}\chi \bigoplus_{j=|\frac{n}{2}|}^{\infty} \mathcal{H}^\chi_j \simeq L^2(SL(2,\mathbb{C})) \,, \tag{2.43}$$

[9]We are writing the Hodge operator next to the Immirzi parameter, unlike what we did in Chapter 1. This is to make contact with the literature, and it has no fundamental impact in what follows.

where $\mathcal{H}_j^\chi$ denotes the space spanned by the $\varphi_{j,m}^\chi$ functions, irreducible under the right regular representation.

For the case at hand, this whole discussion serves the purpose of justifying an analogue of equation (2.33). Using equation (2.42), it is immediate to see that the Dirac delta decomposes as

$$\delta(g) = \frac{1}{2}\sum_n \int \mathrm{d}p\,(n^2+p^2)\sum_{j,m}(D^*)^\chi_{jmjm}(g)\,. \tag{2.44}$$

This integral is of course badly divergent and needs a regularization, but we will not concern ourselves with this problem right now.

Discretizing the constraints

The problem now is to restrict the quantum states of the model such that they satisfy the constraints in equation (2.40). A particularly simple way of doing so is to consider again the full form of the simplicity constraints, $B^{IJ}\wedge B^{KL} = \epsilon^{IJKL}V$, and extract from it a condition on the algebra elements. By expanding $B^{IJ} = B^{IJ}_{\mu\nu}\,\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu$, the simplicity constraints are equivalent to the identity (no summation on the greek indices)

$$\epsilon_{IJKL}B^{IJ}_{\mu\nu}B^{KL}_{\gamma\delta} = \epsilon_{\mu\nu\gamma\delta}v\,, \tag{2.45}$$

for some $v\in\mathbb{R}$. This equation should now be applied in the context of a discretization of the action. In the same way that we associated holonomies to faces in the case of the 3d model, the 2-forms B are naturally associated to 2-simplices (triangles) of each 4-simplex of a chosen triangulation Δ. We therefore define, for each triangular region r, the objects

$$b_r^{IJ} = \int_{r\subset\Delta} B^{IJ}\,, \tag{2.46}$$

which are elements of the Lie algebra of $SL(2,\mathbb{C})$. We will call these objects *bivectors*, for reasons that will become clear below. The constraints of equation 2.45, although before interpreted as global constraints on the field B, now become constraints on the local b_r bivectors

$$\epsilon_{IJKL}\int_{r,r'} B|_r{}^{IJ}\wedge B|_{r'}{}^{KL} = \int_{r,r'} V\,, \tag{2.47}$$

where the integration is over the affine span of two triangle regions r, r', which may or not be disjoint. This allows for three distinct cases [15]:

1. $r = r'$: $\epsilon_{IJKL}b_r^{IJ}b_r^{KL} = 0\,,$ (2.48)
2. $r\neq r'$ and $r\cap r'\neq\emptyset$: $\epsilon_{IJKL}b_r^{IJ}b_{r'}^{KL} = 0\,,$ (2.49)
3. $r\neq r'$ and $r\cap r'=\emptyset$: $\epsilon_{IJKL}b_r^{IJ}b_{r'}^{KL}\neq 0\,,$ (2.50)

differing by the fact that in the first two cases the affine span is of a lower dimension than the form to be integrated. In what follows we will disregard condition number 3, as it turns out it is dynamically imposed [30]. It furthermore turns out that the first two constraints (2.48),(2.49) admit a geometrical interpretation [31] that simplifies their formulation.

Indeed, note that equation (2.48) is a simplicity condition on the IJ indices of the bivectors b_r^{IJ} (this is the reason why we called them bivectors). The proof is straightforward and follows the reasoning of the analogous one in subsection 1.3.2. Each bivector can then be written as an antisymetrized product of vectors $b_r = e \wedge e'$ or its Hodge dual $b_r = \star(e \wedge e')$, and since each bivector corresponds to a triangle region, we may think of each elementary vector e, e' as an edge of that triangle, with the area of that triangle given by the norm of b_r. Furthermore, the second equation asserts that only three of the four vectors constituting $b_r, b_{r'}$ are linearly independent. The first two constraints (2.48) and (2.49) then become the following geometrical conditions

1. Each b_r is simple, *i.e.* it is the exterior product of two vectors or its Hodge dual,
2. The planes defined by bivectors $b_r, b_{r'}$ sharing an edge span a 3-dimensional space.

Note that, together with a closure condition $\sum_{r \in t} b_r = 0$, these constraints are enough to define a non-singular tetrahedron t. Using this geometrical interpretation, a condition that specifies to either sector of the simplicity constraints can be constructed. We have two important lemmas from [31]:

Lemma 2.4.1. A bivector b_r^{IJ} in $\mathbb{R}^4$ is simple if and only if there exists a third vector n^I such that $b_r^{IJ} n_J = 0$, which holds if and only if $(\star b_r)^{IJ} n_J = 0$.

Lemma 2.4.2. Two simple bivectors $b_r^{IJ}, b_{r'}^{IJ}$ span a 3-dimensional subspace of $\mathbb{R}^4$ if and only if there exists a vector n^I such that $b_r^{IJ} n_J = 0 \ \wedge \ b_{r'}^{IJ} n_J = 0$, which holds if and only if $(\star b_r)^{IJ} n_J = 0 \ \wedge \ (\star b_{r'})^{IJ} n_J = 0$.

Note that the lemmas above hold equivalently whether we use b_r or $\star b_r$, but as remarked in [31] this equivalence is not valid if one considers more than a pair of bivectors. That is to say that, while each of the constraints

$$\begin{cases} n_I b_r^{IJ} = 0, & \text{for all } r \text{ in the same tetrahedron } t \\ n_I (\star b_r)^{IJ} = 0, & \text{for all } r \text{ in the same tetrahedron } t \end{cases} \tag{2.51}$$

is enough to enforce equations (2.48) and (2.49), they cannot hold for the same tetrahedron at the same time for clear dimensional reasons. A choice of one of the constraints is then a specialization of the aforementioned equations to a particular sector. The implementation of the constraints $n_I (\star b_r)^{IJ} = 0, \forall r \in t$ in an $SL(2, \mathbb{C})$ spin-foam model is known as the Engle-Pereira-Rovelli-Livine model [32] (henceforth EPRL), and this is the construction we will follow[10].

[10]Note that, since we are considering the Holst action (2.40), both the constraints would be valid choices (under an eventual redefinition of the Immirzi parameter).

Next we must translate this condition into a restriction on the states of the theory. Since the variable conjugate to A in (2.40) is associated to the bivector $\tilde{r}_f^{IJ} = \left(1 + \frac{1}{\gamma}\star\right) b_r^{IJ}$, we start by inverting this equation to find

$$b_r^{IJ} = \frac{\gamma^2}{\gamma^2+1}\left(1 - \frac{1}{\gamma}\star\right)\tilde{b}_r^{IJ} \,. \tag{2.52}$$

Each bivector is an element of the algebra $\mathfrak{sl}(2,\mathbb{C})$, and therefore we may impose the constraints on the generators J_{IJ} of the group rather than on the components of $b_r = b_r^{IJ} J_{r,IJ}$. The second equation of (2.51) then becomes a constraint on the generators, and it reads

$$\begin{aligned} n_I \star \left(1 - \frac{1}{\gamma}\star\right) J_r^{IJ} &= \delta_I^0 \left(\epsilon^{IJ}{}_{KL} J_r^{KL} + \frac{1}{\gamma} J_r^{IJ}\right) \\ &= \frac{1}{2}\epsilon^{0j}{}_{kl} J_r^{kl} + \frac{1}{\gamma} J_r^{0j} \,, \end{aligned} \tag{2.53}$$

where in the first equality we have specified to the case $n_I = \delta_I^0$ as in [32], and the lower-case roman letters take non-zero values. This is a very fortunate combination of the generators, because the terms appearing in the last equation are the well-known boost and rotation generators $K^i = J^{0i}$ and $L^i = \epsilon_{ijk} J^{jk}/2$ of $SL(2,\mathbb{C})$. The condition of equation (2.51) can now be written in its final form as

$$C_r^j = L_r^j + \frac{1}{\gamma} K_r^j = 0 \,. \tag{2.54}$$

To find the true Hilbert space of the theory, one would then consider the subspace that is simultaneously annihilated by all the C_r^j. It turns out, unfortunately, that the constraint C_f^j found above cannot be imposed directly on the states of the model, the reason being that the algebra of constraints does not close [30], as one sees from the commutator

$$[C_r^i, C_{r'}^j] = 2\delta_{rr'}\epsilon^{ij}{}_k C_r^k - \delta_{rr'}\frac{\gamma^2+1}{\gamma^2}\epsilon^{ij}{}_k L_r^k \,. \tag{2.55}$$

Clearly then, unless one sets γ as to make the additional term vanish or specifies $L_r = 0$, one cannot strongly impose $C_r^j \left|\psi\right\rangle = 0$ for all j.

Implementing the constraints

There is more than one way to implement some weaker version of the C_t^j constraints, but the original EPRL proposal is to consider a classicaly equivalent *master constraint*, defined as

$$M_r = C_{r,i} C_r^i = \frac{\gamma^2+1}{\gamma^2} L_r^2 - \frac{1}{\gamma^2} C_{1,r} - \frac{2}{\gamma} C_{2,r} \,, \tag{2.56}$$

where C_1, C_2 denote the Casimirs of $SL(2,\mathbb{C})$, given in terms of K, L by [33][11]

$$\begin{aligned} C_1 = L^2 - K^2\,, \quad & C_1\,|(n,p);j,m\rangle = \frac{1}{4}(n^2-p^2-4)\,|(n,p);j,m\rangle \\ C_2 = -L\cdot K\,, \quad & C_2\,|(n,p);j,m\rangle = \frac{1}{4}pn\,|(n,p);j,m\rangle\ , \end{aligned} \tag{2.57}$$

and look for its smallest positive eigenvalue[12] (as was to be expected, the master constraint does not have the zero eigenvalue in its spectrum in the space $\gamma, p \in \mathbb{R}$, $j \in \mathbb{N}/2$ and $n \in \mathbb{Z}$, as one can check). The master constraint acts on the states as

$$\gamma^2 M\,|(n,p);j,m\rangle = \left[\frac{1}{4}(p^2-n^2) - \frac{1}{2}\gamma pn + (\gamma^2+1)j(j+1) + 1\right]|(n,p);j,m\rangle\ , \tag{2.58}$$

with the requirement from the representation theory of $SL(2,\mathbb{C})$ that $j \geq |n/2|$.

We then want to minimize $\lambda(p,n,j) = p^2 - n^2 - 2\gamma pn + 4(\gamma^2+1)j(j+1) + 4 > 0$ in the space mentioned above. From $\partial_p \lambda = 0$ we find $p = \gamma n$, a condition which must be satisfied at the minimum, and substituting back we find

$$\begin{aligned} \lambda(\gamma n, n, j) &= (\gamma^2+1)[4j(j+1) - n^2] + 1 \\ &\geq (\gamma^2+1)\left[2|n|\left(\frac{|n|}{2}+1\right) - n^2\right] + 1 \\ &\geq 2(\gamma^2+1) + 1\,, \end{aligned} \tag{2.59}$$

where in the second line we used the smallest possible $j = |n|/2$ and in the last line $n = \pm 1$. We have just found the minimum of λ, attained for the particular combination $j = 1/2, n = \pm 1$. To find which other combinations of j, m allow λ to attain that minimum, we simply plug back $\lambda(\gamma n, n, j) = 2(\gamma^2+1) + 1$, obtaining $n^2 = 4j(j+1) - 2$. This has no solutions beside the previously found one, so the usual procedure is now to take the approximation of large spin j, and thus get the solutions

$$\begin{cases} n \simeq 2j \\ p \simeq 2j\gamma\,. \end{cases} \tag{2.60}$$

The $SL(2,\mathbb{C})$ representations satisfying these equations are usually refered to as *simple* representations in the literature.

We thus see that the chosen constraints restrict the states $|(n,p);j,m\rangle$ to be of the form $|(n,\gamma n); n/2, m\rangle$ with $n > 2$ (j must be a half-integer) and, in accordance with the basis (2.41), these are indeed the states with the smallest possible j for a given n. Additionaly, one sees that the previously-continuous label p now has quantized values in proportion to

[11]Note that in the cited reference the author is using as the p, n parameters as half of our own p, n parameters, and this is the reason for the difference in the appearence of the eigenvalues of the Casimirs.

[12]Although the function looks like it can go to negative infinity by taking n large enough, one needs to remember that j is at least half the absolute value of n.

the Immirzi parameter. Finally, note that then we have an isomorphism I of Hilbert spaces (recall that a basis for $\mathcal{H}^\chi$ is given by (2.41)),

$$\begin{aligned} I : L^2(SL(2,\mathbb{C}))|_{\text{constr.}} &\to L^2(SU(2)) \\ |(\gamma n, n); n/2, m\rangle &\mapsto |n/2, m\rangle \end{aligned} \tag{2.61}$$

$$\begin{aligned} L^2(SU(2)) &\to L^2(SL(2,\mathbb{C}))|_{\text{constr.}} \\ |j, m\rangle &\mapsto |(2j\gamma, 2j); j, m\rangle \ , \end{aligned} \tag{2.62}$$

in essence implying that for every function on $SL(2,\mathbb{C})$ that can be constructed from the restricted states there exists a unique $SU(2)$ function, and *vice-versa*. Moreover we have projector maps $\Gamma_\gamma : L^2(SL(2,\mathbb{C})) \to L^2(SL(2,\mathbb{C}))|_{\text{constr.}} \simeq L^2(SU(2))$,

$$\begin{aligned} \Gamma_\gamma : L^2(SL(2,\mathbb{C})) &\to L^2(SU(2)) \\ D^{(p,n)}_{j_1 m_1 j_2 m_2}(g) &\mapsto D^{n/2}_{m_1 m_2}(u) \, , \end{aligned} \tag{2.63}$$

where $u \in SU(2) \subset SL(2,\mathbb{C})$. There is also an induced embedding depending on the Immirzi parameter,

$$\begin{aligned} \Gamma^\dagger_\gamma : L^2(SU(2)) &\to L^2(SL(2,\mathbb{C})) \\ D^j_{m_1 m_2}(u) &\mapsto D^{(2j\gamma,2j)}_{j m_1 j m_2}(g) \, . \end{aligned} \tag{2.64}$$

The physical Hilbert space

It follows from the preceding discussion of Section 2.2 that the Hilbert spaces of the theory should be defined on the boundary Σ of spacetime regions R. As discussed in the first section of this chapter, we expect spin-network states on the space $L^2(SL(2,\mathbb{C})^{|\mathcal{E}_\phi|}/SL(2,\mathbb{C})^{|\mathcal{V}|_\phi})$ for an embedded graph ϕ on Σ. However, according to subsection 2.3.3, we would also like the graph associated to these states to be defined by the structure of the dual complex Δ^* of a spacetime triangulation Δ. In terms of the edges $\mathcal{E}$, vertices $\mathcal{V}$ and faces $\mathcal{F}$ of $\Sigma^* \subset \Delta^*$, we then expect the Hilbert spaces

$$h_\Sigma = \bigoplus_{(\chi,j)\to\mathcal{F}} \bigotimes_{e\in\mathcal{E}} \mathrm{Inv}_{SL(2,\mathbb{C})} \left(\bigotimes_{f\in\mathcal{S}_e} \mathcal{H}_f \bigotimes_{f\in\mathcal{T}_e} \mathcal{H}^*_f \right) \tag{2.65}$$

for the unconstrained theory, where $(\chi, j) \to \mathcal{F}$ denotes a labeling of $SL(2,\mathbb{C})$ representations for each face. These are the Hilbert spaces we must now constrain, and to the resulting physical Hilbert space we will call H_Σ. The EPRL choice is to define H_Σ as

$$H_\Sigma = \bigoplus_{j\to\mathcal{F}} \bigotimes_{e\in\mathcal{E}} \mathrm{Inv}_{SU(2)} \left(\bigotimes_{f\in\mathcal{S}(e)} \mathcal{H}^{(2j_f\gamma,2j_f)}_{j_f} \bigotimes_{f\in\mathcal{T}(e)} \mathcal{H}^{*(2j_f\gamma,2j_f)}_{j_f} \right) \, , \tag{2.66}$$

which we now comment on. Note that the solution of the constraints was found above by specifying a form for the normal vector $n^I = \delta^I_0$ at every tetrahedron in equation (2.53), corresponding to setting all tetrahedra to be space-like with respect to the internal metric η_{IJ}. However, this vector is clearly not invariant under the action of the full $SL(2,\mathbb{C})$ group, being only invariant under some $SU(2)$ subgroup, so we cannot expect the states to be invariant with respect to the initial gauge symmetry. Geometrically, we may indeed think of the possible choices of normal time-like vectors as elements of the upper sheet of the hyperboloid $H^+ = \{x \in \mathbb{R}^{3,1} | x_0^2 - \vec{x}^2 = 1\}$, and one can show $SL(2,\mathbb{C})/SU(2) \simeq H^+$ [34]. The role of the simplicity constraints, unsurprisingly, is then a degree of freedom reduction by breaking some of the symmetry of the theory. While the original EPRL model took the same choice we did for the normal vector, more general constructions are available, making use of the concept of *projected spin-networks*, which allow for a more flexible description of an n^I object at each tetrahedron. On this the reader is directed to the relevant literature, *e.g.* [35].

To arrive at the physical Hilbert space from the unconstrained BF theory, we therefore want to project h_Σ to H_Σ. To this end, consider now the embedding $\Gamma_\gamma^\dagger$ above, and recall the $SU(2)$ projectors as in equation (2.34), but this time in a four-dimensional context (this changes only the number of tensored Hilbert spaces), and denote them $\tilde{\pi}_e$. Consider also the same type of projectors but now for the $SL(2,\mathbb{C})$ Hilbert spaces, which we call π_e. Using the Γ_γ projector, we can construct a map going from a tensor product of $SL(2,\mathbb{C})$ spaces to an invariant subpace under the action of $SU(2)$ by composing $f := \Gamma_\gamma^\dagger \circ \tilde{\pi}_e \circ \Gamma_\gamma$. This is then a map:

$$f : \bigotimes_{f\in\mathcal{S}(e)} \mathcal{H}_{j_f}^{(p_f,n_f)} \bigotimes_{f\in\mathcal{T}(e)} \mathcal{H}_{j_f}^{*(p_f,n_f)} \to \mathrm{Inv}_{SU(2)} \left(\bigotimes_{f\in\mathcal{S}(e)} \mathcal{H}_{j_f}^{(2j_f\gamma,2j_f)} \bigotimes_{f\in\mathcal{T}(e)} \mathcal{H}_{j_f}^{*(2j_f\gamma,2j_f)} \right) . \tag{2.67}$$

The projection from h_Σ to the physical space can therefore be achieved simply by taking $f(h_\Sigma)$. Since the unconstrained spin-model defines the space h_Σ precisely through a set of projectors π_e, we may define our new model by choosing, in the spin-foam partition function, the assignment $\pi_e \to \pi_e \circ f \circ \pi_e$ (we choose this instead of $\pi_e \to f \circ \pi_e$ simply for symmetry reasons).

Putting everything together

Now we can finally write down the spin-foam partition function for the EPRL model. We start as we did in 3d with the unconstrained BF theory,

$$Z(M) = \int \mathcal{D}A\mathcal{D}B \; e^{i\int_M \mathrm{Tr}(F[A]\wedge B)} \tag{2.68}$$

$$= \int \mathcal{D}A \; \delta(F[A]) \, , \tag{2.69}$$

where now M is four-dimensional and the fields take values in the $\mathfrak{sl}(2,\mathbb{C})$ algebra. As before, we also introduce an oriented triangulation Δ inducing a dual complex Δ^*. We denote the set of faces and edges of the dual complex by $\mathcal{F}, \mathcal{E}$, respectively, and we write down the same discretized action as above

$$Z(\Delta^*) = \int \prod_{e\in\mathcal{E}} \mathrm{d}g_e \prod_{f\in\mathcal{F}} \delta\left(\prod_{e\in\partial f} g_e\right) , \tag{2.70}$$

where now crucially the group integration is over $SL(2,\mathbb{C})$. Now we consider the expansion of the Dirac delta of equation (2.44). Using the notation (note that this trace is different from the trace defined for the Fourier transform of $SL(2,\mathbb{C})$ functions above)

$$\mathrm{Tr}[D^\chi] := \sum_{jm} D^\chi_{jmjm} , \tag{2.71}$$

and droping the normalization factor in the expansion of the delta, we find

$$\begin{aligned}
Z(\Delta^*) &= \int \prod_{e\in\mathcal{E}} \mathrm{d}g_e \prod_{f\in\mathcal{F}} \left(\sum_{n\in\mathbb{Z}} \int_{\mathbb{R}} \mathrm{d}p\ (n^2+p^2)\mathrm{Tr}\left[\prod_{e\in\partial f} (D^*)^\chi (g_e)\right]\right) \\
&= \sum\!\!\!\!\!\!\int_{\chi\to\mathcal{F}} \int \prod_{e\in\mathcal{E}} \mathrm{d}g_e \prod_{f\in\mathcal{F}} \left((n^2+p^2)_f \mathrm{Tr}\left[\prod_{e\in\partial f} D^*_f (g_e)\right]\right) \\
&= \sum\!\!\!\!\!\!\int_{\chi\to\mathcal{F}} \left[\prod_{f\in\mathcal{F}} (n^2+p^2)_f\right] \int \prod_{e\in\mathcal{E}} \mathrm{d}g_e \mathrm{Tr}_{f\in\mathcal{F}} \left[\prod_{f\in\mathcal{F}} \left(\prod_{e\in\partial f} D^*_f (g_e)\right)\right] \\
&= \sum\!\!\!\!\!\!\int_{\chi\to\mathcal{F}} \left[\prod_{f\in\mathcal{F}} (n^2+p^2)_f\right] \mathrm{Tr}_{f\in\mathcal{F}} \left[\prod_{e\in\mathcal{E}} \left(\int \mathrm{d}g_e \prod_{f:\, e\in\partial f} D^*_f (g_e)\right)\right] ,
\end{aligned}$$

where we have followed the exact same steps used before for the three-dimensional theory, and by $(n^2+p^2)_f$ we mean the sum using the assigned values (n,p) to a face f. The trick of interchanging the sum with the product, which we have used very frequently, is in this case considerably less justified. However, since in the end the p labels will be quantized and its integral will reduce to a sum, we assume we can proceed.

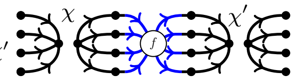
The argument of the trace is again a projection map for each edge,

$$\begin{aligned}
\pi_e : \bigotimes_{f\in\mathcal{S}(e)} \mathcal{H}^*_{D_f} \bigotimes_{f\in\mathcal{T}(e)} \mathcal{H}_{D_f} &\to \mathrm{Inv}\left(\bigotimes_{f\in\mathcal{S}(e)} \mathcal{H}^*_{D_f} \bigotimes_{f\in\mathcal{T}(e)} \mathcal{H}_{D_f}\right) \\
\pi_e &= \int_{SL(2,\mathbb{C})} \mathrm{d}g_e \bigotimes_{f\in\mathcal{S}(e)} D^*(g_e) \bigotimes_{f\in\mathcal{T}(e)} D_f(g_e) ,
\end{aligned} \tag{2.72}$$

and here we can implement the simplicity constraints using the map of equation (2.67). To do so, we need to replace each of the above projectors π_e with the constrained ones, and this can be done by composing each projector of the unconstrained spin-foam with f. Since f is made up of the projecting maps Γ_γ, the inclusion of f at every edge also collapses the integral and sum over (p, n) labels to a sum over j labels. Using $\Delta_{(p,n)} = (p^2 + n^2)$, the partition function then takes the form

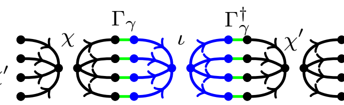
$$Z(\Delta^*) = \sum\!\!\!\!\!\!\int_{\chi\to\mathcal{F}} \left[\prod_{f\in\mathcal{F}} (n^2+p^2)_f\right] \mathrm{Tr}_{f\in\mathcal{F}} \left[\prod_{e\in\mathcal{E}} \pi_e \circ f \circ \pi_e\right]$$

$$= \sum_{j\to\mathcal{F}} \left[\prod_{f\in\mathcal{F}} j_f^2(\gamma^2+1)\right] \mathrm{Tr}_{f\in\mathcal{F}} \left[\prod_{e\in\mathcal{E}} \int \mathrm{d}\chi\, \mathrm{d}\chi'\, \Delta_\chi \Delta_{\chi'} \right.$$



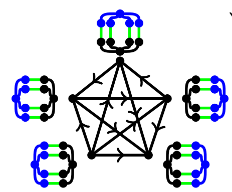
$$\left.\right]$$

$$= \sum_{j\to\mathcal{F}} \left[\prod_{f\in\mathcal{F}} j_f^2(\gamma^2+1)\right] \mathrm{Tr}_{f\in\mathcal{F}} \left[\prod_{e\in\mathcal{E}} \sum_{\iota} \int \mathrm{d}\chi\, \mathrm{d}\chi'\, \Delta_\chi \Delta_{\chi'} \right.$$



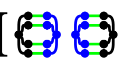
$$\left.\right]$$

$$= \sum_{j\to\mathcal{F}} \left[\prod_{f\in\mathcal{F}} j_f^2(\gamma^2+1)\right] \left[\prod_{v\in\mathcal{V}} \left(\prod_{i=1}^{5} \left[\sum_{\iota_i} \int \mathrm{d}\chi_i \Delta_{\chi_i}\right] \right.\right.$$

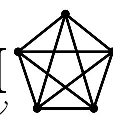


$$\left.\left.\right)\right]$$

$$= \sum_{j\to\mathcal{F}} \sum_{\iota\to\mathcal{E}} \int \mathrm{d}\chi_i \Delta_{\chi_i} \left[\prod_{f\in\mathcal{F}} j_f^2(\gamma^2+1)\right] \left[\prod_{e\in\mathcal{E}} \right.$$



$$\left.\right] \left[\prod_{v\in\mathcal{V}} \right.$$



$$\left.\right] .$$

In the second line we used that each edge of a 4-simplex is shared by four faces, and throughout omitted the normalization diagrams for simplicity of the presentation. We represent the $SL(2,\mathbb{C})$ projector in black in terms of the Clebsch-Gordan coefficients. In the third line we have expanded the f map in terms of the $SU(2)$ Clebsch-Gordan in blue and the green projectors Γ_γ, whose matrix coefficients are usually called *fusion coefficients* in the literature. In the fourth line we have contracted the indices of the projectors around each face, and in this way finding an $SL(2,\mathbb{C})$ $10j$-symbol for each vertex, and five numbers related to the fusion coefficients. In the last line we translated the sum over $SU(2)$ intertwiners to the beginning of the expression. The integration over $SL(2,\mathbb{C})$ was also pushed to the left and written in a simplified way to declutter the notation. The amplitude associated to each vertex spin-network finally reads

$$A\left(\partial F|_R\right) = \int \mathrm{d}\chi_i \Delta_{\chi_i} \left[\prod_{f\in(\mathcal{F}\cap R^*)} j_f^2(\gamma^2+1)\right] \left[\prod_{e\in\mathcal{E}} f(\iota,\chi) f^*(\iota,\chi)\right] \left[\prod_{v\in(\mathcal{V}\cap R^*)} 10j(\chi)\right] , \tag{2.73}$$

where now each minimal spin-foam $\partial F|_R$ is made up of a 2-complex Δ^* with $SU(2)$ j-representation labels on the faces and ι-intertwiner labels on the edges. By f we denote the numbers constructed from the fusion coefficients, and the $SU(2)$ and $SL(2,\mathbb{C})$ intertwiners. The $10j$ symbol is between the constrained representations, and also depends on the choice of a continuous χ label for each intertwiner.

Having discussed the model, let us now take a step back and comment on the construction we presented. Starting from a constrained topological lagrangian for gravity (1.25), classically equivalent to the Einstein-Hilbert theory in the absence of matter, we have defined a way of assigning amplitudes to spin-network states associated to the boundaries of spacetime regions. The amplitude was derived from a path-integral/sum-over-histories construction, and the constraints imposed after quantization. While the final expression (2.73) for the amplitude is divergent due to the non-compact nature of $SL(2, \mathbb{C})$, having the need for a regularization procedure, the amplitude is still non-perturbative in nature. For these reasons the model seems to faithfully implement the expectations of subsection 1.1.4 for a quantum theory of gravity, up to a complete background independence (note that the amplitude assigned to the state seems to depend on the particular triangulation one chooses). On this subject we note that a full spin-foam amplitude is expected to be found only after some procedure for removing the dependence on the triangulation is carried out, for example by a weighted summing over every possible dual-complex consistent with the boundary structures. We mention that the very promising group field theory framework [36] was developed, among other reasons, to tackle precisely this problem.

It is also worthwhile to remark that the boundary Hilbert spaces defined by the EPRL model have a very similar structure to the ones one deals with in the loop quantum gravity approach, there being the expectation that this framework can be understood as a covariant formulation of LQG. It is likely that the next step in the maturation of the theory will be in the direction of understanding the continuum and classical limits of the model, such that a correspondence to conventional GR at lower energies can be established. This remains a difficult problem.

Chapter 3

Causality Considerations

Having discussed in the previous chapter the EPRL model, itself a *bona fide* theory of quantum gravity, we now turn to a collection of considerations regarding the possible causal structure we may associate to spin-foam models. We start by introducing the general framework of quantum causal histories, and, after proposing a notion of causality on a spin-foam, we establish a correspondence between both frameworks. Because we expect spin-foams to generate a great number of causal loops, we also investigate the behavior of evolution operators in the presence of those loops.

3.1 Quantum causal histories

Between the many possible models studied for quantum gravity, the causal sets approach and derivative frameworks remain some of the most popular. In this section we introduce the structure of these models.

3.1.1 The causal sets proposal

Although in physical analysis of general relativity one is not frequently worried about the particular topology with which the spacetime manifold M is defined, it turns out [37] that one may choose a topology for M that in some sense encodes important proprieties of that spacetime, *e.g.* its causal structure. In the article just mentioned such a topology is constructed, called the *path topology*, defined in the following way:

Definition 3.1.1. Let (M, g) be a smooth 4-dimensional Lorentzian manifold without boundary, and $\mathcal{M}$ the topology on M. The path topology $\mathcal{P}$ is defined to be the finest topology in M such that the induced topology on time-like curves coincides with the original topology $\mathcal{M}$.

Soon after this new topology for spacetime was proposed, a very strong result which referred to it was proven in [38] by Malament. It turns out that the following fact holds:

Theorem 3.1.1 (Malament)**.** If (M, g) is a smooth 4-dimensional Lorentzian manifold without boundary and $f : (M, g) \to (N, h)$ is a homeomorphism with respect to the path topology, then f is a smooth conformal isometry.

Here conformal isometry means an isometry up to a conformal transformation *i.e.* a homeomorphism inducing the association $g \mapsto \Omega^2 h$ with $\Omega \in C^\infty(M)$.

This is the inspiration to the *causal set* approach to quantum gravity (a good review is [39]). The hope is that one is able to recover every possible information about spacetime solely from knowing both the causal relations enforced by that spacetime and how to assign a volume to fix the unknown conformal factor. These informations are easy to categorize if one furthermore internalizes the assumption of a discrete spacetime (much like the spin-foam approach does). A causal set, then, is a *partially-ordered* set (or *poset*) C with a relation $\succeq$ between some of its elements. The elements of the set are taken to be akin to spacetime events, while the partial ordering is associated to a directed causal relation between them. If x is in the future of y then we write $x \succeq y$. The relation $\succeq$ must satisfy:

1. Transitivity: $\forall\, x, y, z \in C, \;\; x \succeq y \wedge y \succeq z \Rightarrow x \succeq z$.

   If an event is causally related to a second, and the second is causally connected to a third, then the first and the third must also be causally related.

2. Antisymmetry: $\forall\, x, z \in C, x \succeq y \wedge y \succeq x \Rightarrow x = y$.

   There should be no closed causal loops: an event cannot be in the future and in the past of another.

3. Local finiteness: $\forall\, x, y \in C, \;\; |\{z \in C | x \succeq z \succeq y\}| < \infty$.

   There should only be a finite number of events between the past and the future of two chosen events.

4. Reflexivity: $\forall\, x \in C, \;\; x \succeq x$.

   An event is causally connected to itself.

It turns out that finite partially ordered sets admit a representation as an oriented graph, called the *Hasse diagram*. To draw it, one assigns vertices to elements of the set and ordered edges between them such that they represent the *shortest* causal relation, that is, if there is an edge from a vertex x to a vertex y then there does not exist any z such that $y \succ z \succ x$.

The causal set program seeks to extract physics from this extremely simple framework. Causal sets approximate Lorentzian manifolds by referring to Malamant's theorem: let the causal relations of events be described by the poset, and fix the conformal factor with the natural volume measure that comes from counting subset elements. One way to look at this correspondence is to consider a sprinkling of the causal set on a Lorentzian manifold in accordance with its causality structure. One usually considers a Poisson embedding $\Phi : C \to M$ such that the probability of finding n elements in a volume v is

$$P(n, v) = \frac{(\rho v)^n}{n!} e^{-\rho v}\,, \tag{3.1}$$

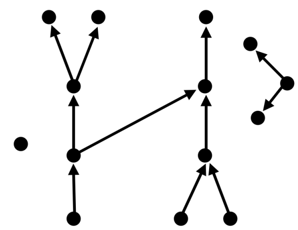


Figure 3.1: An example of a causal set.

and, as aimed for, the expectation value relates to the volume as $\langle n \rangle \sim \rho v$ for large n. The problem of how to extract a *single* continuum spacetime is still an open question in the causal sets theories however, although the expectation is that this is possible.

3.1.2 Adding quantum structure

The fact that the causal set approach is dependent on what is essentially a graph that can be embedded on a manifold seems to suggest a possible relation with spin-networks and spin-foams. With this in mind, a modification to the causal set framework was proposed by Markoupolou in [40]. In much the same way that spin networks and spin-foams encode algebraic data besides the combinatorial one, Markopoulou's proposal consists exactly in appending Hilbert spaces and quantum states to causal sets.

One would like to define unitary operators between the Hilbert spaces at the vertices of the causal set graph, interpreted as an evolution of the states from one Hilbert space to the other. To this end, consider for example the left $\mathcal{H}_l$ and right $\mathcal{H}_r$ Hilbert spaces at both ends of the diagonal middle arrow in Figure 3.1. Since the direction of the graph is to be interpreted as a causal relation, one should expect the states in $\mathcal{H}_r$ to be related to the states in $\mathcal{H}_l$, but also to the states below $\mathcal{H}_r$. However, a unitary operator from $\mathcal{H}_l$ to $\mathcal{H}_r$ would uniquely determine all the states in $\mathcal{H}_r$ solely from the states in $\mathcal{H}_l$, without any influence from states in the vertex below $\mathcal{H}_r$. A more reasonable construction, therefore, would be to consider unitary operators not between vertices, but between sets of vertices constituting a complete past and a complete future of some event. To make this explicit, we state first the following definitions:

1. The *causal past* P of an event x is the set of all events to the past of x, *i.e.* $P(x) = \{y \in C \,|\, x \succeq y\}$. The *causal future* F is analogously defined.

2. An *acausal subset* $A \subset C$ is a subset of C where all elements are pairwise unrelated.

3. An acausal set A is a *complete past* of x if every element of $P(x)$ is related to an event in A. The *complete future* is analogously defined.

4. A *complete pair* is a pair of acausal sets A, B such that A is a complete future for every element in B and B is a complete past for every element in A.

Now, the definition of complete pair clearly induces another relation on the causal set, but this time between acausal subsets of C. If an acausal subset A is a complete future of the acausal subset B and B is a complete past of A we write $A \succeq B$ to indicate they form a complete pair. In this way we construct an induced poset of acausal sets. It is in this poset that we will define Hilbert spaces, formulating the framework of *quantum causal histories*:

1. To each acausal set $A \in C$ we associate a Hilbert space $\mathcal{H}_A$. Given that all the elements in A are unrelated they can be taken as isolated systems. We demand $H_A = \bigotimes_{x \in A} \mathcal{H}_x$.

2. Information must be conserved by construction between each complete pair, so to each complete pair of acausal sets $B \succeq A$ we associate a unitary evolution operator $U_{A,B} : \mathcal{H}_A \to \mathcal{H}_B$. We must demand that dim($\mathcal{H}_A$) =dim($\mathcal{H}_B$).

3. Transitivity implies that, if $B \succeq A$ and $C \succeq B$, we must have $U_{A,C} = U_{B,C} \circ U_{A,B}$.

Finally, there is *a priori* no good reason to why we should consider the Hilbert spaces to be associated to vertices and operators to edges and not *vice-versa*. Indeed, if we want vertices of the causal set to be understood as space-time events, then assigning the Hilbert spaces to the edges of the graph and operators to the vertices would be more in-line with the physical prescriptions we have been supporting in this work. Interpreting each node as a spacetime event, hypothetically not defined as a 0-dimensional object but rather as a representation of an elementary spacetime region, the edges of the partial ordering adjacent to that vertex have then a natural interpretation in terms of the boundary of that region. Thus we make contact with the GBF framework if we assign quantum states to the edges of the poset, rather than to the vertices, and this is the approach we will take in what follows. This prescription has the added benefit of making it considerably easier to assign operators to the causal set, as it is immediate to see that the set of incoming edges at a vertex and the set of outgoing ones already constitutes a complete pair of acausal sets, so evolution operators can indiscriminately be assigned to every vertex in the set. We thus modify the definition of a QCH as follows:

1. To each edge e in the graph of C we associate a Hilbert space $\mathcal{H}_e$.

2. Information must be conserved between every complete pair, so to each element $x \in C$ we associate an operator $U_{e,e'} : \bigotimes_{\text{inc.}} \mathcal{H}_e \to \bigotimes_{\text{outg.}} H_{e'}$.

3. Transitivity implies that the vertex operators can be composed.

3.2 A consistency condition from causality

In this section we discuss a consistency condition arising from the interplay between quantum mechanics and a spacetime with time-like loops. This condition was proposed initially by Deutsch in [41], and implemented to some extent on the framework of quantum causal histories in [42]. Here we attempt to expand on this previous work.

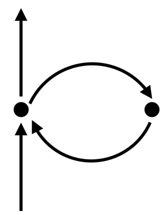


Figure 3.2: A simple QpCH with a causality loop and two vertex operators.

3.2.1 Including causal loops in QCH

Since QCHs are defined over a causal set, causal loops are not immediately described by the framework. Indeed, one of the properties of the relation associated to causal sets, as discussed above, is the one of *antisymmetry*, which prohibits such structures. Given that many approaches to quantum gravity (and indeed even classical gravity) naturally allow closed time-like loops, one may try to describe them in this framework by relaxing the demand for that property. To this end, we define a pseudo-causal set, for the lack of a better name, by

- A set $\mathcal{C}$ together with a binary relation $\succeq$ between the elements, satisfying
  - Reflexivity: $a \succeq a$
  - Transitivity: $a \succeq b \wedge b \succeq c \Rightarrow a \succeq c$
  - Local finiteness: $\forall a, c \in \mathcal{C}, |\{b \in \mathcal{C} | a \succeq b \succeq c\}| < \infty$

  (a relation that satisfies the first two proprieties is called a *preordering*, and a set together with a relation satisfying the above three is called a *chronal set*.)

Just as before, any pseudo-causal set can be denoted by a directed graph, but this time not necessarily acyclic, since any preordered set can be described by such a graph. To the relation of the set we call a pseudo-causal relation, and the definitions of causal future and past carry over to the ones of pseudo-causal future and past. We endow now this pseudo-causal set with the structure of a quantum history by assigning Hilbert spaces to the edges and operators to the vertices, as above, generating a quantum pseudo-causal history (QpCH). An example of a QpCH with a causal loop is given by Figure 3.2.

3.2.2 The Deutsch condition

A very simple consistency condition for the application of quantum mechanics in the presence of causal loops was proposed early by Deutsch. Here we review the argument. Consider the setting of Figure 3.3, that is, a system initially described by some Hilbert space $\mathcal{H}_A$ propagating along a flat $1+1$ spacetime where the dashed lines have been identified. This particular type of spacetime goes by the name of *Deutsch-Politzer spacetime*, having been studied in [43, 41], and it roughly corresponds to a Minkowski spacetime with a handle on

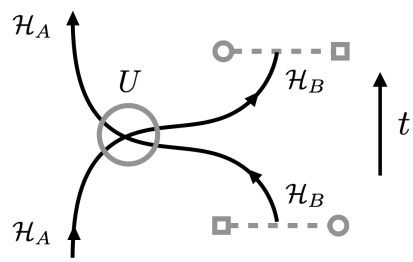


Figure 3.3: Politzer spacetime with a propagating particle. The system interacts with an older version of itself in some finite region, and this interaction is described by U.

top. At sufficiently early times, the particle is described by a state in one Hilbert space $\mathcal{H}_A$, and we assume for simplicity that this state is pure, with density operator ρ_A. After some time there will be two copies of the particle, each one having different proper times, and we describe the states of the older particle with $\mathcal{H}_B$, such that the whole system between the line segments is described by $\mathcal{H}_A \otimes \mathcal{H}_B$. In the region where the trajectories of both particles intersect we expect an interaction described by a unitary operator $U : \mathcal{H}_A \otimes \mathcal{H}_B \to \mathcal{H}_A \otimes \mathcal{H}_B$. Denoting the initial state in $\mathcal{H}_B$ by ρ_B (which might be, in general, mixed), the final state after the interaction will be $U(\rho_A \otimes \rho_B)U^\dagger$. The partial trace over $\mathcal{H}_A$ of this state must be the state in $\mathcal{H}_B$ after the interaction, but because of the structure of the spacetime we are considering this state must also be ρ_B. We then have the consistency condition

$$\mathrm{Tr}_A[U(\rho_A \otimes \rho_B)U^\dagger] = \rho_B \,, \tag{3.2}$$

implying that the states of the system inside the causal loop are not independent of the ones outside.

This system has a very natural formulation in terms of a QpCH. Indeed, it is described succinctly by the history of Figure 3.2, where the left vertex is assigned the U operator and the right vertex is just the identity. To extract more information from this condition, we follow [42] in studying equation (3.2) in components. Assuming the Hilbert spaces in question are finite-dimensional of dimension d, a basis for the Hilbert spaces can be given in terms of $SU(d)$ generators σ^i and the identity matrix σ^0. A general density matrix is then expanded as $\rho = \frac{1}{d}[\sigma^0 + \alpha_i \sigma^i] := \frac{1}{d}\alpha_\mu \sigma^\mu$. To consider a more general case, we require only of the operator U to be a completely positive map, since these are the most general maps one can consider such that a density matrix is mapped to a density matrix. We do not demand that the map is trace-preserving. Using the well-known Kraus decomposition of completely positive maps, we can write

$$\begin{gathered} U : \mathcal{H}_A \otimes \mathcal{H}_B \to \mathcal{H}_A \otimes \mathcal{H}_B \\ U(\rho) = \sum_n K_n \rho K_n^\dagger \,, \quad \sum_n K_n K_n^\dagger \leq 1 \end{gathered} \tag{3.3}$$

where each K_n is a homomorphism $K_n : \mathcal{H}_A \otimes \mathcal{H}_B \to \mathcal{H}_A \otimes \mathcal{H}_B$, usually called a Kraus operator. Denoting the action of K_n by $K_n \sigma_A^\alpha \otimes \sigma_B^\beta K_n^\dagger = (s_n)^{\alpha\beta}_{\mu\nu} \sigma_A^\mu \otimes \sigma_B^\nu$, equation (3.2) then reads

$$\begin{aligned}\frac{1}{d}[\beta_\mu\sigma_B^\mu] &= \sum_n \mathrm{Tr}_A\left[\left(\alpha_\alpha\beta_\beta(s_n)^{\alpha\beta}_{\mu\nu}\right)\frac{\sigma_A^\mu\otimes\sigma_B^\nu}{d^2}\right]\\ &= \frac{\delta_0^\mu\sigma_B^\nu}{d}\sum_n\left[(s_n)^{00}_{\mu\nu}+\alpha_i(s_n)^{i0}_{\mu\nu}+\beta_i(s_n)^{0i}_{\mu\nu}+\alpha_i\beta_j(s_n)^{ij}_{\mu\nu}\right]\\ &= \frac{1}{d}\sum_n\left[\alpha_\mu\beta_\nu(s_n)^{\mu\nu}_{00}\sigma_B^0+\alpha_\mu\beta_\nu(s_n)^{\mu\nu}_{0i}\sigma_B^i\right]\,,\end{aligned}$$

and from orthogonality of the generators we must have that each multiplicative factor on the left equals the corresponding one on the right. Using $\sigma_i\sigma^i = 0$ we have that $\mathrm{Tr}\left[\alpha_\alpha\beta_\beta(s_n)^{\alpha\beta}_{\mu\nu}\sigma_A^\mu\sigma_B^\nu\right] = \alpha_\mu\beta_\nu(s_n)^{\mu\nu}_{00}$, so the previous equation implies the following system:

$$\begin{cases}\mathrm{Tr}\left[U(\rho)\right]=1\\ \beta_j\left[\delta_i^j - s_{0i}^{0j}-\alpha_l s_{0i}^{lj}\right] = s_{0i}^{00}+\alpha_j s_{0i}^{j0}\,,\end{cases} \tag{3.4}$$

where we defined $s^{\alpha\beta}_{\mu\nu} = \sum_n (s_n)^{\alpha\beta}_{\mu\nu}$. Note that the existence of a causal loop constrains the operator to be trace-preserving.

Now, as noted in [42], this last equation can be used to derive more information about the form of the operator U. Indeed, representing by ρ'_A the state of the system at sufficiently large times, we can analogously write

$$\rho'_A = \mathrm{Tr}_B[U(\rho_A\otimes\rho_B)U^\dagger]\,, \tag{3.5}$$

which in components reads

$$\rho'_A = \frac{1}{n}[s^{00}_{\mu 0}+\alpha_i s^{i0}_{\mu 0}+\beta_i s^{0i}_{\mu 0}+\alpha_i\beta_j s^{ij}_{\mu 0}]\sigma_A^\mu\,. \tag{3.6}$$

The second equation (3.4) is in general a linear one constraining β_i to be a rational function of α_i, but this implies from equation (3.6) that the final state of the system is given by a non-linear function of its initial state. If we insist on upholding the principle of linearity of quantum mechanics on both the causality-respecting region and the causal-loop one, we must demand that the β terms in equation (3.6) vanish, as must so to the α terms in equation (3.4). That is, we must assume $s^{0i}_{l0} = s^{ij}_{l0} = s^{lj}_{0i} = s^{j0}_{0i} = 0$. This in turn implies that the only non-zero components of s will be the $s^{0\nu}_{0\mu}$ and $s^{\nu 0}_{\mu 0}$, meaning that the operator U must decouple into two, $U = A\otimes B$. Furthermore, because of the structure of the causal loop we must clearly have $B = \mathbb{1}_{\mathcal{H}_B}$, so our result is that the causal structure and quantum mechanical linearity demand from the vertex operator to have the factorized form

$$U = A\otimes\mathbb{1}_{\mathcal{H}_B}\,. \tag{3.7}$$

3.2.3 Generalized Deutsch condition on cycles

We would now like to generalize the argument of the previous section to an arbitrary cycle, where the arrows do not all have the same orientation, with an arbitrary number of vertices. Before we tackle this problem it is crucial to note that, from the identity $V \otimes W^* \simeq \mathrm{Hom}(V, W)$, and using the associativity of the tensor product, we have the natural correspondence

$$V^* \otimes W^* \otimes U \simeq \mathrm{Hom}(U, V \otimes W) \simeq \mathrm{Hom}(U \otimes W^*, V)\,. \tag{3.8}$$

As a consequence, each operator V_i induces another operator $\tilde{V}_i$ by dualizing some chosen vector spaces. As an example, from the two maps

$$\begin{gathered} V_1 : \mathcal{H}_a \to \mathcal{H}_b \otimes \mathcal{H}_c \otimes \mathcal{H}_d \\ V_2 : \mathcal{H}_c \otimes \mathcal{H}_d \to \mathbb{C} \end{gathered} \tag{3.9}$$

we may construct in a canonical way the operators

$$\begin{gathered} \tilde{V}_1 : \mathcal{H}_d^* \to \mathcal{H}_b \otimes \mathcal{H}_c \otimes \mathcal{H}_a^* \\ \tilde{V}_2 : \mathcal{H}_c \to \mathcal{H}_d^*\,. \end{gathered} \tag{3.10}$$

The first question to ask is what the physical role of this construction is. If a physical system described by the original operators is perfectly equivalent to a physical system described by the induced ones, then the causal relations in a QpCH contribute nothing to the eventual information one might extract from it, and we may freely invert the edge directions where necessary, considering the appropriate dual spaces and induced operators. Here we argue however that the two situations do indeed correspond to different formulations of the problem, both from a conceptual viewpoint and from a physical one.

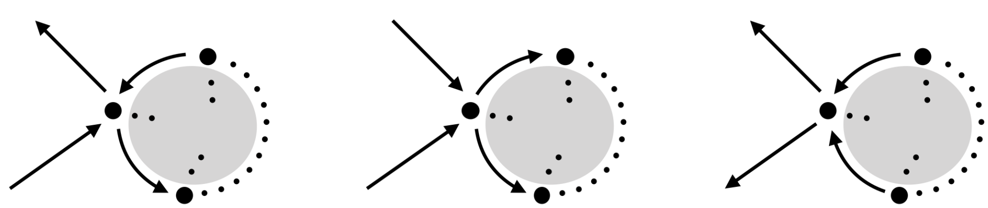


Figure 3.4: Three possible histories with different assignments of edge orientations. The grey blob is meant to represent an arbitrary cyclic structure in the interior of the outside cycle.

Consider then three QpCHs composed of an arbitrary exterior cycle with external edges at only one of the vertices of that cycle, represented in Figure 3.4. The exterior cycle is allowed to contain interior cycles, represented by a grey blob. To edges that only connect to the cycle at one of its ends we will call *acyclic*. The physical situation described by the leftmost history is one of a state in the arbitrary past interacting at some point with a cycle and then evolving to an arbitrary future. We have therefore a notion of a *transition* from a state into another, and the situation is reminiscent of standard quantum mechanics and evolution between time hyper-surfaces. The two remaining histories of that figure, however, tell another story: there the states live at either an arbitrary past or an arbitrary future. In the GBF framework these histories correspond to the same physical picture of states defined on some spacetime boundary, being indeed related by a simple hermitian conjugation of the relevant vertex, physically corresponding to a time inversion. Of course, since all the vertices are connected, consistency demands that the conjugation of a vertex must be followed by conjugation of every other vertex, meaning that a history with only asymptotic incoming edges is equivalent under time reversal to a history with only asymptotic outgoing edges, both characterized by the same amplitude map (eventually up to complex conjugation) and thus containing the same physical information.

From the preceding discussion it follows that we expect that the dualization procedure on the operators, corresponding to an inversion of the edge directions of the history, does necessarily change the physical system if applied to the acyclic edges. However, since, at least for histories where a transition interpretation is available, the closed subgraph corresponds to a transition operator, we may freely invert the edges on that subgraph, dualizing the operators, because this procedure will not change how the operator acts on the acyclic states. Moreover, it is clear now that the generalization of the condition of the last section must be constructed on histories with transitioning acyclic edges, because only there may we demand linearity on the *evolution* of states.

Let us then consider an arbitrary cycle, and focus on a vertex with transitioning acyclic edges. No matter the valency of the vertex (as long as it is larger than 4), by redefining the edge Hilbert spaces as tensor products of edges, and by inverting the directions of the edges inside the cycle, we can always take this particular vertex to the form of figure 3.5.

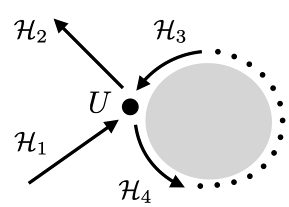


Figure 3.5: A vertex with transitioning edges inside an arbitrary cycle.

We will denote states outside of the causality cycle that are outgoing and incoming by an over-bar $\bar{\rho}^{(+)}, \bar{\rho}^{(-)}$, respectively. Following the same reasoning as above, the out state is given by

$$\bar{\rho}_2^{(+)} = \mathrm{Tr}_4 \left[U \left(\bar{\rho}_1^{(-)} \otimes \rho_3 \right) U^\dagger \right] , \tag{3.11}$$

so it depends on the states defined in both $\mathcal{H}_1$ and $\mathcal{H}_3$. Because of the cycle structure of the diagram, the states in $\mathcal{H}_3$ must satisfy a consistency condition similar to the one discussed in the previous section. Indeed, the state in $\mathcal{H}_3$ can be evolved at U, and taking appropriate traces the resulting states can also be evolved inside the cycle back to the space $\mathcal{H}_3$. This will constrain the state in $\mathcal{H}_3$ to be of the form

$$\rho_3 = f(\rho_3 \otimes \{\bar{\rho}^{(-)}\}) \Rightarrow \rho_3 = f'_{\text{n.l.}}(\{\bar{\rho}^{(-)}\})\,, \tag{3.12}$$

where f is a linear map and $\{\bar{\rho}^{(-)}\}$ denotes all the incoming states outside the cycle. Consequently, the state will have a non-linear dependence $f'_{\text{n.l.}}$ on all the exterior incoming states. Equation (3.11) determines that the outgoing state is a linear function of both the incoming state in $\mathcal{H}_1$ and the cycle state ρ_3. But this implies then, because of equation (3.12), that there is a non-linear mapping

$$\bar{\rho}_2^{(+)} = g_{\text{n.l.}}(\bar{\rho}_1^{(-)}, \{\bar{\rho}^{(-)}\})\,. \tag{3.13}$$

Imposition of linearity at the transition from $\mathcal{H}_1$ to $\mathcal{H}_2$ will thus demand that the non-linear dependence on the second argument is removed, *i.e.* that the operator U must decouple into a tensor product $U = A \otimes B$, where A acts exclusively on the external spaces and B acts on the cycle. Unlike the previous simple situation, however, B will not be an identity map, but rather a complicated operator that directly depends on the structure of the cycle.

Hence we conclude the following: for a QpCH with completely positive maps at the vertices, consistency between causality and linear evolution demands that each external vertex of a cycle must decouple into two maps; one acting on spaces outside the cycle, and one acting on the inside spaces.

3.3 Spin-foams as quantum causal histories

From the preceding discussion it should be clear that the framework of quantum causal histories shares a couple of key basic structures with the spin-foam approach to gravity. In this section we focus on studying to which extent a correspondence between the two frameworks can be established.

3.3.1 Transition amplitudes

One can synthetically describe the spin-foam models discussed in Chapter 2 as a method to assign probability amplitudes to boundary states of spacetime regions. Quite generally, if a history of spacetime is described by a spin-foam $F = (\Delta^*, \rho, \iota)$, then the state associated to the boundary Σ of a region R has a probability amplitude

$$\rho_R(\partial F|_R) = \sum_{\substack{\Lambda_f \to (\mathcal{F}\cap R^*) \\ \Lambda_e \to (\mathcal{E}\cap R^*)}\big|_{\partial F_R}} \left[\prod_{f\in(\mathcal{F}\cap R^*)} W_f(\Lambda_f, \Lambda_e)\right] \left[\prod_{e\in(\mathcal{E}\cap R^*)} W_e(\Lambda_f, \Lambda_e)\right] \left[\prod_{v\in(\mathcal{V}\cap R^*)} W_v(\Lambda_f, \Lambda_e)\right]\,, \tag{3.14}$$

where the notation under the sum indicates a labeling of faces and a labeling of edges by some representation-theoretic data which must agree with the labeling of the boundary state (if R is a minimal spin-foam region the sum disappears, as there is no bulk information to sum-over). For the models discussed in the Chapter 2, characterized by a constrained BF-type theory in n dimensions with symmetry group G and boundary space with $SU(2) \subseteq G$ gauge, the amplitude map is given by

$$\rho_R(\partial F|_R) = \sum_{\hat{\Lambda}_G \to (\mathcal{F}\cap R^*)|_{\partial F_R}} \left[\prod_{f\in(\mathcal{F}\cap R^*)} \dim(\rho_f) \right] \left[\prod_{v\in(\mathcal{V}\cap R^*)} \sum\!\!\!\!\!\!\int_{\chi_1 \cdots \chi_{n+1}} \left(\left\{ \binom{n+1}{2} j \right\}_{(\chi,f)} \prod_{i=1}^{n+1} C^{(\iota_i)}_{a_1 \dots a_n} \Gamma_{a_1 b_1} \dots \Gamma_{a_n b_n} C^{(\chi_k)}_{b_1 \dots b_n} \right) \right] . \tag{3.15}$$

Here $\hat{\Lambda}_G \to (\mathcal{F} \cap R^*)|_{\partial F_R}$ denotes a labeling of faces by a *constrained* choice of irreducible representations of G which agree with the boundary data, the sum/integral is over intertwiner labels of G, the object in the curly brackets is an nj-symbol of G, $C^{(\chi)}$ denotes a labeled G intertwiner and $C^{(\iota)}$ denotes an $SU(2)$ one. Note that the intertwiner labels of $C^{(\iota)}$ are not summed over, but rather determined by the boundary state. The maps Γ are the embeddings (or their hermitian conjugate, where appropriate)

$$\Gamma^\dagger : L^2(SU(2)) \to L^2(G) \tag{3.16}$$

mapping $SU(2)$ states to constrained G ones.

In establishing a correspondence between spin-foam models and quantum causal histories, one must unavoidably encode the above amplitudes in a Hilbert space operator formalism. In accordance with the GBF discussion of Section 2.3, we should expect the model's spin-foam amplitudes to be recovered as transition amplitudes of an operator for situations where it is reasonable to consider some states as incoming and some as outgoing. Since there is a notion of a minimal region of spin-foam models, from a collection of which the whole model is obtained, we further would like to associate the tentative operator to each such region, such that schematically we have the following

$$\langle \text{in}|h|\text{out}\rangle \sim \rho_R(\partial F|_R) \, .$$

While there is no notion of incoming and outgoing states in most spin-foam models (there is rather a probabilistic notion of a "state of affairs"), we can still reasonably define initial and final states in some appropriate Hilbert spaces. This we do in the following.

3.3.2 The edge Hilbert spaces

We shall focus first on one fundamental element of the spin-foam framework, *i.e.* one n-simplex R of the considered triangulation Δ. As was argued in subsection 2.3.3, the dual

2-complex $R^* \subset \Delta^*$ associated to each simplex induces a closed graph on the boundary, and hence, for the class of spin-foams at hand, it induces a boundary Hilbert space

$$H_{\partial R^*} = \bigoplus_{\hat{\Lambda}_G \to \mathcal{F}|_{R^*}} \bigotimes_{e \in \mathcal{E}} \mathrm{Inv}_{\mathrm{SU}(2)} \left(\bigotimes_{f \in \mathcal{S}(e)} \mathcal{H}_f \bigotimes_{f \in \mathcal{T}(e)} \mathcal{H}_f^* \right), \tag{3.17}$$

where $\hat{\Lambda}_G$ denotes the set of *constrained* unitary irreducible representations of the larger symmetry group G and the action of $SU(2)$ is through the bi-regular representation. This Hilbert space is exactly the one defined in equation (2.8). However, in order to have a notion of transition, *i.e.* a concept of incoming states being mapped to outgoing ones, one would like to think not in terms of states associated to the whole boundary of an n-simplex, but rather in terms of states at each face of that simplex. This way we can define Hilbert spaces H_e for each edge $e \in \mathcal{E} \subset R^*$, corresponding to each $(n-1)$-simplex of the fundamental element, such that a choice of one state in each edge corresponds to a choice of a state in $H_{\partial R^*}$[1]. One can then choose from the n spaces a subset of incoming spaces and a subset of outgoing ones.

Looking back to subsection 2.3.3, and in particular to Figure 2.7, it is clear that the dual complex will induce, at each face of the simplex, a graph with one n-valent vertex connected to n 1-valent ones. The full graph on the boundary can then be obtained by gluing the open edges of each of these graphs along the boundary. We are therefore interested in objects that look like spin-networks with some 1-valent vertices, but where intertwiners are *not* assigned to any of those vertices, so that the open edges can be glued together. To these objects we will refer as *open* spin-networks. The prescription for such a gluing, and the definition of such one-vertex states, was already discussed by Oriti in [44], and here we formulate the construction in the spin basis. Assuming the validity of the Peter-Weyl theorem for G, for each edge $e \in \mathcal{E} \subset R^*$ we define the spaces

$$\begin{aligned} \tilde{H}_e &= L^2(G^n) \\ &\simeq \bigoplus_{\hat{\Lambda}_G \to \mathcal{F}|_{R^*}} \left[\left(\bigotimes_{f \in \mathcal{S}(e)} \mathcal{H}_f \bigotimes_{f \in \mathcal{T}(e)} \mathcal{H}_f^* \right) \otimes \left(\bigotimes_{f \in \mathcal{S}(e)} \mathcal{H}_f \bigotimes_{f \in \mathcal{T}(e)} \mathcal{H}_f^* \right)^* \right], \end{aligned} \tag{3.18}$$

as the non-gauge-invariant Hilbert space of a graph composed of one n-valent vertex and n 1-valent vertices. Note that this is in complete agreement with equation (2.6). To recover the gauge-invariant space one would now quotient out the action of two copies of $SU(2) \subset G$. Instead, in order to have open legs on the networks, we define the "half" action

$$L : g_e \mapsto \bigoplus_{\hat{\Lambda}_G \to \mathcal{F}|_{R^*}} \left[\left(\bigotimes_{f \in \mathcal{S}(e)} \rho^f(g_e) \bigotimes_{f \in \mathcal{T}(e)} \rho^{f*}(g_e) \right) \otimes \mathbb{1} \right], \tag{3.19}$$

[1] Crucially, note that the total boundary space cannot be written simply as the tensor product of spaces of intertwiners for each edge, because of the first direct sum in the definition of $H_{\partial R^*}$.

which can be thought of as half a gauge transformation of a parallel transport, acting only in one of the ends of its curve. We then obtain the edge Hilbert space as the quotient

$$\begin{gathered} H_e \simeq \tilde{H}_e/\sim \\ |j_1\alpha_1\beta_1,\, \dots, j_n\alpha_n\beta_n\rangle \sim L(g_e)\, |j_1\alpha_1\beta_1,\, \dots, j_n\alpha_n\beta_n\rangle\ , \forall g_e \in SU(2) \subset G\,. \end{gathered} \tag{3.20}$$

In other words, we have the characterization

$$\begin{aligned} H_e &= \bigoplus_{\hat{\Lambda}_{G\to\mathcal{F}}|_{R^*}} \left[\mathrm{Inv}_{SU(2)} \left(\bigotimes_{f\in\mathcal{S}(e)} \mathcal{H}_f \bigotimes_{f\in\mathcal{T}(e)} \mathcal{H}_f^* \right) \otimes \left(\bigotimes_{f\in\mathcal{S}(e)} \mathcal{H}_f \bigotimes_{f\in\mathcal{T}(e)} \mathcal{H}_f^* \right)^* \right] \\ &\simeq \bigoplus_{\hat{\Lambda}_{G\to\mathcal{F}}|_{R^*}} \left[\mathrm{Int}_G^\Gamma \left(\bigotimes_{f\in\mathcal{S}(e)} \mathcal{H}_f,\, \bigotimes_{f\in\mathcal{T}(e)} \mathcal{H}_f \right) \otimes \left(\bigotimes_{f\in\mathcal{S}(e)} \mathcal{H}_f \bigotimes_{f\in\mathcal{T}(e)} \mathcal{H}_f^* \right)^* \right], \end{aligned} \tag{3.21}$$

and Int_G^Γ denotes the space of G intertwiners coming from $SU(2)$ ones through the embedding map $\Gamma^\dagger$. A convenient label for the basis states is the ket $|j_1 m_1,\, \dots, j_n m_n;\, \iota\rangle$, where ι denotes an $SU(2)$ intertwiner, j labels the constrained representations of G and m is the magnetic index of the second factor in the above equation.

Now we discuss the gluing procedure. Every closed spin-network state can be obtained by a linear combination of edge states through

$$|j_1^{(1)} \dots j_n^{(1)}, \iota_1; \dots ; j_1^{(n+1)} \dots j_n^{(n+1)}, \iota_{n+1}\rangle = \sum_{\{m\}_1 \dots \{m\}_{n+1}} |\{j,m\}_1, \iota_1\rangle\ \dots\ |\{j,m\}_{n+1}, \iota_{n+1}\rangle\, \{\delta_m\}\ ,$$

where $\{\delta_m\}$ denotes a product of Kronecker deltas ensuring coherence between the edges that are to be glued. As a practical example of this procedure, consider two open spin-networks with a 3-valent vertex that are coherent, i.e. they have the same edge representations and consistent orientations, as in Figure 3.6.

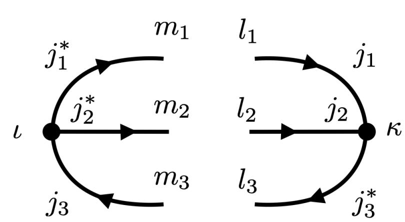


Figure 3.6: Two coherent open spin-networks labeled by intertwiners at the vertices, representations at the edges, and magnetic indices at the open vertices.

Each of these spin-networks can be represented as a ket or as an $SU(2)$ function as

$$\begin{gathered} |j_1^* m_1, j_2^* m_2, j_3 m_3; \iota\rangle \to a_{m_3 m_2 m_1}(g_1, g_2, g_3) = \iota_{a_3 a_2 a_1} D^{(j_3)}_{a_3 m_3}(g_3) D^{*(j_2)}_{a_2 m_2}(g_2) D^{*(j_1)}_{a_1 m_1}(g_1) \\ |j_1 l_1, j_2 l_2, j_3^* l_3; \kappa\rangle \to b_{l_1 l_2 l_3}(h_1, h_2, h_3) = \kappa_{b_1 b_2 b_3} D^{(j_1)}_{b_1 l_1}(h_1) D^{(j_2)}_{b_2 l_2}(h_2) D^{*(j_3)}_{b_3 l_3}(h_3)\,, \end{gathered} \tag{3.22}$$

and the closed spin-network state resulting from the gluing is given by

$$
\begin{aligned}
&\sum_{\{m\}\{l\}} |j_1^* m_1, j_2^* m_2, j_3 m_3; \iota\rangle\, |j_1 l_1, j_2 l_2, j_3^* l_3; \kappa\rangle\, \delta_{m_1 l_1}\delta_{m_2 l_2}\delta_{m_3 l_3} \\
&\to \iota_{a_3 a_2 a_1} D^{(j_3)}_{a_3 m_3}(g_3) D^{*(j_2)}_{a_2 m_2}(g_2) D^{*(j_1)}_{a_1 m_1}(g_1) \kappa_{b_1 b_2 b_3} D^{(j_1)}_{b_1 m_1}(h_1) D^{(j_2)}_{b_2 m_2}(h_2) D^{*(j_3)}_{b_3 m_3}(h_3) \\
&= \iota_{a_3 a_2 a_1} D^{(j_3)}_{a_3 b_3}(g_3 h_3^{-1}) D^{(j_2)}_{b_2 a_2}(h_2 g_2^{-1}) D^{(j_1)}_{b_1 a_1}(h_1 g_1^{-1}) \kappa_{b_1 b_2 b_3}\,,
\end{aligned}
$$

which is the gauge-invariant function we expect from a state in $H_{\partial R^*}$.

3.3.3 The quantum causal history of a spin-foam

The necessary ingredients to establish a possible correspondence between a given spin-foam model and the quantum causal histories framework have now been laid down. First of all, the framework of QCH must be relaxed to that of QpCH, defined in subsection 3.2.1, to accommodate for causal loops. The correspondence is then established as follows: the pseudo-causal set C of the QpCH will be composed of every vertex $v \in \Delta^*$ of the dual triangulation, as the "skeleton" of the spin-foam, and an ordering of C will be defined through the oriented edges $e \in \Delta^*$ in the natural way, *i.e.* $v_1 \in \mathcal{S}(e) \wedge v_2 \in \mathcal{T}(e) \Rightarrow v_2 \succeq v_1$. The set C is then upgraded to a QpCH by assigning to each element of the set what we will call a *history operator*, defined as a map

$$
\begin{aligned}
h : \bigotimes_{e \in \mathcal{T}(v)} H_e &\to \bigotimes_{e \in \mathcal{S}(v)} H_e \\
\langle \text{in}|h|\text{out}\rangle &= \rho_R(\partial F|_R)\,, \qquad (3.23)
\end{aligned}
$$

where R denotes a minimal spin-foam region (*i.e.* an n-simplex), such that to each edge in the graph of C there corresponds the Hilbert space H_e of equation (3.21). We demand moreover from whichever particular spin-foam model is under consideration:

1. That each $(n-1)$-simplex of the model be space-like;
2. That the history operator be unitary;
3. That the history operator satisfy, at least weakly, the Deutsch condition on cycles (subsection 3.2.3).

Note that these three conditions are necessary for a consistent notion of causality for spin-foams. In order to understand the orientations on the edges of Δ^* as a causal ordering we must reasonably demand that those edges are associated with a time direction. Under this identification, unitarity follows from requiring a well-behaved quantum theory[2]. If the edges are understood as causal relations, and since the spin-foams naturally incorporate a large number of causal loops through the faces in the dual triangulation, we must demand

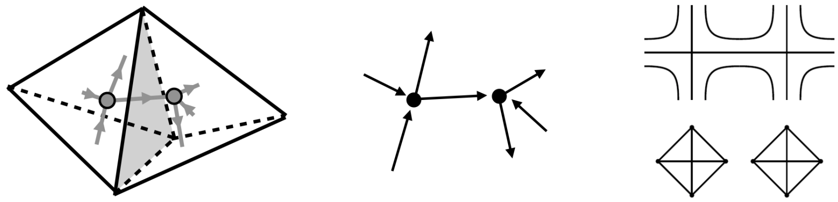


Figure 3.7: A QCH of riemannian $3d$ BF theory, obtained by gluing two tetrahedra together, and the matrix coefficients of the resulting composite operator.

consistency with quantum mechanics through the Deutsch condition (under the requirement of linear evolution). We would like to explicitly point out that in understanding the orientations on the edges of a spin-foam as a causal ordering we are making the key physical assumption that such a causal ordering can be implemented at the quantum level. In our view, however, it would also seem reasonable to argue that causality could be expected to be solely an effective concept: if, as it is hoped, one could extract an effective metric from a quantum state, this metric would already by itself encode an effective causal structure. Since the problem of the continuum limit for spin-foam theories is still unresolved we proceed with our identification, in the hope that a specification at the quantum level of a causal structure might help in restricting the possible effective configurations associated to the quantum theory.

Finally, note that generic spin-foam history operators are not expected to conform to the Deutsch condition strongly, the reason being that the structure of the operator very much encodes the combinatorial nature of the spin-foam construction. As we will show below in the context of the models discussed in this work, the history operator must ensure that the states assigned to each $(n-1)$-simplex on the boundary are coherent with one-another, such that a spin-network on the boundary is properly defined. Consequently, the history operator will not in general reduce to a tensor product of operators as the condition demands. One would benefit from constructing an appropriate notion of "weak" Deutsch condition which could be demanded from spin-foam models, for example through the imposition on a semi-classical approximation. Alternatively one could also forfeit the requirement of linearity in state transitions, allowing for a more complex behavior of the dynamics of quantum states. Nonlinear quantum models have been extensively used in many areas in physics, even if only in the context of effective models, so one has to consider the possibility that a quantum mechanical formulation of gravity might demand such an uncommon behavior.

[2] Of course, one might consider more general ways of implementing unitarity, for example as some effective behavior when a sum over triangulations is considered.

3.4 The history operator for EPRL-type models

Now we focus on explicitly describing and studying the history operator for the models discussed in Chapter 2, namely the three-dimensional constrained BF theory and the four-dimensional EPRL model.

3.4.1 Explicit construction

To start with, note from equation (3.21) that the boundary Hilbert space of these models (3.17) cannot simply be written as the product of edge Hilbert spaces H_e, as indeed we have artificially enlarged the degrees of freedom. The states of the boundary Hilbert space will only correspond to those products of edge states that are coherent with one-another, *i.e.* gluings of one-vertex graphs that share leg representations and orientations. The vertex operator we want to define must therefore ensure this coherence, which ultimately is a consequence of the combinatorial structure of the underlying triangulation one is working with, by assigning vanishing amplitudes to non-agreeing states. We therefore define for each spin-foam vertex, given some choice of edge and face orientations, the family of *history operators*

$$h : \bigotimes_{e\in\mathcal{T}(v)} H_e \to \bigotimes_{e\in\mathcal{S}(v)} H_e \,, \tag{3.24}$$

associated to each simplex of Δ, and therefore each vertex of Δ^*. The matrix coefficients of these operators, *independently of the domain-codomain structure*[3], are chosen to be

$$\begin{aligned} h_{\{j,m\}_1,\iota_1;\dots\{j,m\}_{n+1},\iota_{n+1}} &= \sum\!\!\!\!\!\!\int_{\chi_1\cdots\chi_{n+1}} \left(\left\{ \binom{n+1}{2} j \right\}_{(\chi,j)} \prod_{i=1}^{n+1} C^{(\iota_i;\{j\}_i)}_{a_1\dots a_n} \Gamma_{a_1b_1}\dots\Gamma_{a_nb_n} C^{(\chi_i;\{j\}_i)}_{b_1\dots b_n} \right) \\ &\quad \cdot \prod_{l=1}^{\frac{n(n+1)}{2}} \frac{\delta_{(j\in\{j\}_i,j'\in\{j\}_{k\neq i})}}{\dim(j)} \delta_{(m\in\{m\}_i,m'\in\{m\}_{k\neq i})} \,, \end{aligned} \tag{3.25}$$

where each δ in the product is between pairs of representations associated to different edges. Note that the deltas constrain configurations to have a non-vanishing amplitude only when the choice of states at each face of the simplex actually corresponds to a spin-network on the boundary. In order to fully recover the spin-foam amplitude, however, one must still sum over the magnetic indices; because there is a larger space of states at the boundary, we must identify a special subset of states at each H_e, which we call *boundary-like states*, given by a sum over the magnetic label

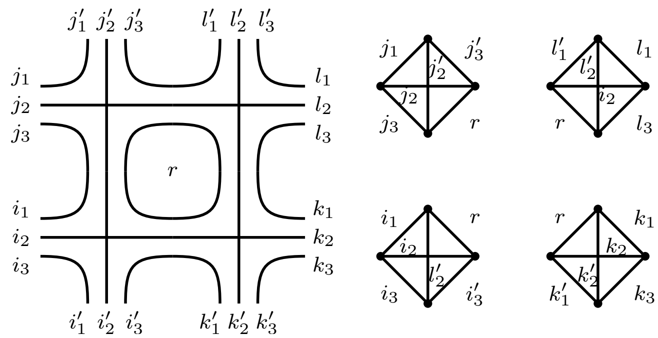
$$H_e \ni |\{j\},\iota\rangle_{\text{b.l.}} = \sum_{\{m\}} |\{j,m\},\iota\rangle \,, \tag{3.26}$$

[3] Crucially, the coefficients must be the same for every choice of domain-codomain structure because the spin-foam amplitudes for the models discussed in this work are invariant under a change of edge orientation, as argued in the end of subsection 2.4.1.

such that the spin-foam amplitudes correspond to transition amplitudes between such states at boundary edges. This sum operates the gluing of the states of H_e into a spin-network state on the full boundary.

Having a larger set of states at the boundary of each n-simplex is also useful to generate spin-foam face amplitudes by simple composition of the history operators. Composition corresponds in coefficients to matrix multiplication, which includes a sum over the basis that defines the components, and therefore contributes from the Kronecker deltas a dimension factor. As an example, consider the $3d$ BF history operator h of equation (3.29) and the following composition:

$$(h \circ h \circ h \circ h)_{\{j\}\dots} = \sum_{\tau_1\tau_2\tau_3\tau_4} \sum_{r}$$



$$= \sum_{\lambda \to \mathcal{F}} \left[\prod_{f \in (\mathcal{F} \cap R^*)} \dim(\rho_f) \right] \left[\prod_{v \in (\mathcal{V} \cap R^*)} 6j \;\; \delta_{f_1 f_1'} \dots \delta_{f_6 f_6'} \right] . \qquad (3.27)$$

This particular choice induces a closed circle diagram labeled by the representation r, corresponding to a face amplitude $\dim(r) = 2r + 1$. The Kronecker deltas insure the boundary states are induced from representations on each face $f \in \mathcal{F} \subset \Delta^*$, so that a choice of a state in one face of a tetrahedron constrains the possible choices of states in every neighboring face. In this way the usual face amplitudes are recovered.

Lastly we show explicit examples of the construction we have been presenting, by specifying to both Riemannian BF theory in $3d$ (2.36) and the EPRL model 2.4.2:

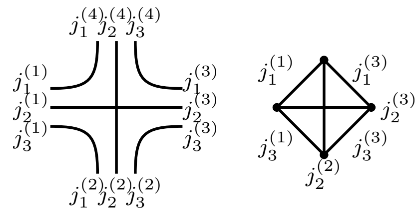
- **Riemannian $3d$ BF model**

  We take $G = SU(2)$ and no constraints on the states. Choose for concreteness the domain-codomain structure

$$h : \mathcal{H}_{e_1} \otimes \mathcal{H}_{e_2} \to \mathcal{H}_{e_3} \otimes \mathcal{H}_{e_4} \, , \qquad (3.28)$$

  where we have not written the intertwiner labels in the states, because they are in this case uniquely defined. Omitting the diagrammatic normalization of the $6j$ symbol, the line orientations and the dimension terms, the coefficients have the form:

$$ {}_{\{j,m\}_4;\{j,m\}_3}h_{\{j,m\}_1;\{j,m\}_2} = \tag{3.29}$$



$$ = \frac{\delta^{m_1^{(1)} m_1^{(4)}}_{j_1^{(1)} j_1^{(4)}} \delta^{m_2^{(1)} m_3^{(3)}}_{j_2^{(1)} j_2^{(3)}} \delta^{m_3^{(1)} m_1^{(2)}}_{j_3^{(1)} j_1^{(2)}} \delta^{m_1^{(3)} m_3^{(4)}}_{j_1^{(3)} j_3^{(4)}} \delta^{m_3^{(3)} m_3^{(2)}}_{j_3^{(3)} j_3^{(2)}} \delta^{m_2^{(4)} m_2^{(2)}}_{j_2^{(4)} j_2^{(2)}}}{\dim(j_1^{(1)})\dim(j_2^{(2)})\dim(j_1^{(3)})\dim(j_3^{(1)})\dim(j_3^{(2)})\dim(j_2^{(1)})} \begin{Bmatrix} j_1^{(1)} & j_2^{(2)} & j_1^{(3)} \\ j_3^{(1)} & j_3^{(3)} & j_2^{(3)} \end{Bmatrix} . $$

A spin-foam amplitude is obtained by selecting physical states in each H_e,

$$ \begin{aligned} A(\{j\}_1; \dots \{j\}_4) &= \sum_{\{m\}_3\{m\}_4} \langle \{j,m\}_3, \iota_3| \langle \{j,m\}_4, \iota_4| \, h \sum_{\{m\}_1\{m\}_2} |\{j,m\}_1, \iota_1\rangle |\{j,m\}_2, \iota_2\rangle \\ &= \sum_{\{m\}_1\{m\}_2\{m\}_3\{m\}_4} {}_{\{j,m\}_4;\{j,m\}_3}h_{\{j,m\}_1;\{j,m\}_2} \\ &= \begin{Bmatrix} j_1^{(1)} & j_2^{(2)} & j_1^{(3)} \\ j_3^{(1)} & j_3^{(3)} & j_2^{(3)} \end{Bmatrix} . \end{aligned} \tag{3.30} $$

- **EPRL model**

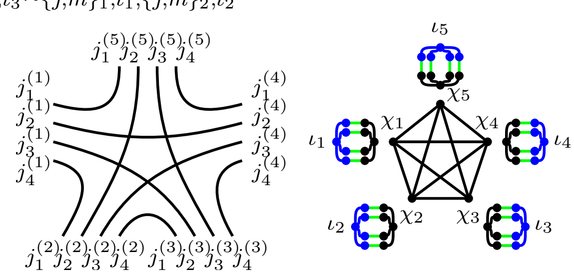
  We take $G = SL(2,\mathbb{C})$ and demand that λ runs over only the constrained $SL(2,\mathbb{C})$ representations. We again make a concrete choice for the structure of the operator,

$$ h : \mathcal{H}_{e_1} \otimes \mathcal{H}_{e_2} \to \mathcal{H}_{e_3} \otimes \mathcal{H}_{e_4} \otimes \mathcal{H}_{e_5} \, , \tag{3.31} $$

  and once more omit the diagrams' normalizations and dimension factors. The coefficients should then read:

$$ {}_{\{j,m\}_5,\iota_5;\{j,m\}_4,\iota_4;\{j,m\}_3,\iota_3}h_{\{j,m\}_1,\iota_1;\{j,m\}_2,\iota_2} = $$

$$ = \int \left[\prod_{i=1}^{5} \mathrm{d}\chi_i \Delta_{\chi_i} \right] $$

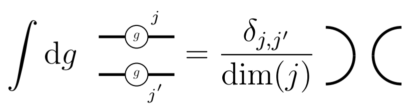


$$ = \frac{\delta^{m_1^{(1)} m_1^{(5)}}_{j_1^{(1)} j_1^{(5)}} \dots \delta^{m_4^{(4)} m_4^{(3)}}_{j_4^{(4)} j_4^{(3)}}}{\dim(j_1^{(1)}) \dots \dim(j_4^{(4)})} \int \left[\prod_{i=1}^{5} \mathrm{d}\chi_i \Delta_{\chi_i} \right] \{10j\}_{(\chi,j)} \prod_{i=1}^{5} \left[C^{(\iota_i;\{j\}_i)}_{a_1 \dots a_n} \Gamma_{a_1 b_1} \dots \Gamma_{a_n b_n} C^{(\chi_i;\{j\}_i)}_{b_1 \dots b_n} \right] . \tag{3.32} $$

where the the integral over χ_l refers to the intertwiners of $SL(2,\mathbb{C})$, and $\Delta_\chi = (n^2 + p^2)$ for $\chi = (p,n)$. Repeated indices in the second square brackets of the last line are summed-over, and C^{ι}, C^{χ} represent $SU(2)$ and $SL(2,\mathbb{C})$ intertwiners, respectively. Again, the spin-foam amplitude is recovered as a transition amplitude between physical boundary states.

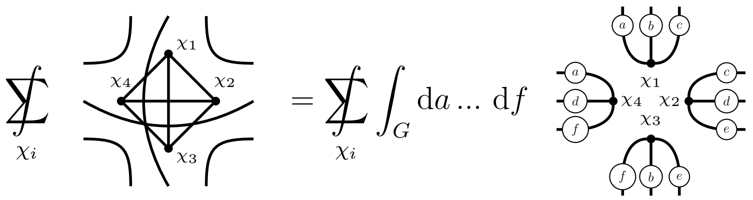
3.4.2 An alternative characterization

We can rewrite the components of the history operator 3.25 in a form that more explicitly reveals the underlying spin-foam structure. Consider first the diagrammatic Schur identity of section B.2,



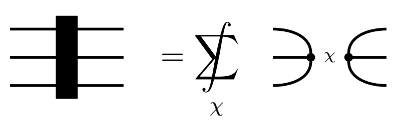
(3.33)

where the labeled lines indicate j-representation matrices of g. Then, focusing first on an history operator in 3 dimensions without fusion coefficients, we have



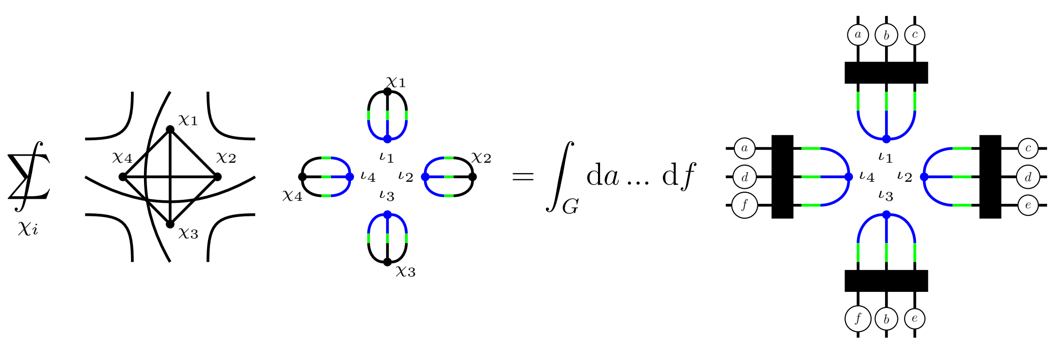
(3.34)

where the diagram on the right stands for representation matrices contracted with G intertwiners. The right-hand diagram automatically produces the dimension factors that we omitted on the left-hand one. Now the inclusion of the fusion coefficients can be made, which also allows us to hide the sum/integral over intertwiner labels. Representing the projectors onto the invariant subspace by a solid bar,



(3.35)

we have the equality



(3.36)

Explicitly, the general history operator can therefore be written in the following way: define for each edge the coefficients

$$\begin{aligned} &E(g_1,\, \dots, g_n)_{\{j\},\iota;\, a_1, \dots a_n} \\ &= C^{(\iota,\{j\})}_{d_1 \dots d_n} \Gamma_{d_1 c_1} \dots \Gamma_{d_n c_n} \left[\int_G \mathrm{d}h D^{j_1}_{c_1 b_1}(h) \dots D^{j_n}_{c_n b_n}(h) \right] D^{j_1}_{b_1 a_1}(g_1) \dots D^{j_n}_{b_n a_n}(g_n)\,, \end{aligned} \tag{3.37}$$

such that, with respect to these, the history operator reads

$$h_{\{j,m\}_1;\iota_1 \dots \{j,m\}_{n+1};\iota_{n+1}} = \int_G \left[\prod_{l=1}^{\frac{n(n+1)}{2}} \mathrm{d}g_l \right] \prod_{i=1}^{n+1} E(\sigma_i)_{\{j\},\iota;\, a_1, \dots a_n} \tag{3.38}$$

where σ_i are non-repeating choices of subsets with n elements from the set $\{g_i\}_{i=1,\dots,n(n+1)/2}$ such that each element of the larger set is chosen only twice. Note that the above diagrammatic notion visually encodes the notion that the spin-foam amplitude is obtained by summing over the magnetic indices, as one can find the amplitude by simply joining the black lines that are labeled by the same letter.

3.4.3 The matter of unitarity

The mathematical theory of infinite-dimensional Hilbert spaces and their operators is beyond the scope of this work, but some comments can still be made regarding the nature of the history operator defined in equations (3.24) and (3.25), and their respective spaces of definition (3.21).

First, we note that it is a well-known result from the theory of L^p spaces that $L^2(G)$ is a separable Hilbert space if and only if G is second-countable and locally compact (see *e.g.* [45]). Since the groups we are working with ($SL(2,\mathbb{C})$ for the 4-dimensional model, and $SU(2)$ for the 3-dimensional toy-model) satisfy these two conditions, and in equation (3.20) we quotient out a closed subspace, the resulting edge Hilbert space H_e is still separable[4]. Since all infinite-dimensional separable Hilbert spaces are isomorphic, this implies that the domain and codomain of definition of the history operator in equation (3.24) are isomorphic to each other. Thus, there being a different number of tensor product factors in the domain and codomain of the operator is not an obstruction *per se* to unitarity. It does turn out, however, that for the class of models under discussion the history operator cannot be unitary.

To see why it is so, consider for simplicity two history operators of the 3-dimensional BF model with the domain-codomain structures

$$\begin{aligned} h_{3,1} &: H_e \otimes H_e \otimes H_e \to H_e \\ h_{2,2} &: H_e \otimes H_e \to H_e \otimes H_e\,. \end{aligned} \tag{3.39}$$

[4] Many thanks to Maximilian H. Ruep for both pointing this out to me and helping with the following argument.

Now we assume $h_{3,1}$ is unitary, and hence it must map an orthonormal basis to an orthonormal basis. Defining $c_{1,2,3}$ to be the norm of the image of a basis state under $h_{3,1}$,

$$\begin{aligned} c_{1,2,3} &:= ||h_{1,3}\,|\{j,m\}_1,\iota_1\rangle\,|\{j,m\}_2,\iota_2\rangle\,|\{j,m\}_3,\iota_3\rangle\,||^2 \\ &= \sum_{\{j,m\}_4,\iota_4} |\,\langle\{j,m\}_4,\iota_4|\,h_{1,3}\,|\{j,m\}_1,\iota_1\rangle\,|\{j,m\}_2,\iota_2\rangle\,|\{j,m\}_3,\iota_3\rangle\,|^2 = 1\,, \end{aligned}$$

we have that the norm of the image of $h_{2,2}$ satisfies

$$\begin{aligned} ||h_{2,2}\,|\{j,m\}_1,\iota_1\rangle\,|\{j,m\}_2,\iota_2\rangle\,||^2 &= \sum_{\substack{\{j,m\}_3,\iota_3\\ \{j,m\}_4,\iota_4}} |\,\langle\{j,m\}_3,\iota_3|\,\langle\{j,m\}_4,\iota_4|\,h_{2,2}\,|\{j,m\}_1,\iota_1\rangle\,|\{j,m\}_2,\iota_2\rangle\,|^2 \\ &= \sum_{\substack{\{j,m\}_3,\iota_3\\ \{j,m\}_4,\iota_4}} |\,\langle\{j,m\}_4,\iota_4|\,h_{1,3}\,|\{j,m\}_1,\iota_1\rangle\,|\{j,m\}_2,\iota_2\rangle\,|\{j,m\}_3,\iota_3\rangle\,|^2 \\ &= \sum_{\{j,m\}_3,\iota_3} c_{1,2,3} \sim \infty\,. \end{aligned}$$

We thus see that $h_{2,2}$ cannot be unitary. But then neither can $h_{1,3}$, because

$$\begin{aligned} ||h^\dagger_{1,3}\,|\{j,m\}_4,\iota_4\rangle\,||^2 &= \sum_{\{j,m\}_{1,2,3},\iota_{1,2,3}} |\,\langle\{j,m\}_1,\iota_1|\,\langle\{j,m\}_2,\iota_2|\,\langle\{j,m\}_3,\iota_3|\,h^\dagger_{1,3}\,|\{j,m\}_4,\iota_4\rangle\,|^2 \\ &= \sum_{\{j,m\}_{1,2,3},\iota_{1,2,3}} |\,\langle\{j,m\}_4,\iota_4|\,h_{1,3}\,|\{j,m\}_1,\iota_1\rangle\,|\{j,m\}_2,\iota_2\rangle\,|\{j,m\}_3,\iota_3\rangle\,|^2 \\ &= \sum_{\{j,m\}_2,\iota_2} ||h_{2,2}\,|\{j,m\}_1,\iota_1\rangle\,|\{j,m\}_2,\iota_2\rangle\,||^2 \sim \infty\,, \end{aligned}$$

so we arrive at a contradiction. The same argument can be applied to any operator in the family.

Note that in the argument we did not need to use the explicit form of the coefficients of the history operator; only that those coefficients should be the same independently of the underlying domain-codomain structure of the operator. In turn, this is a consequence of the fact that the spin-foam amplitude of the EPRL model (and related ones) does not depend on the orientations of the dual-complex which induce that structure.

Given this behavior of the history operator, the first relevant remark we can make is that the spin-foam models of the type we have been considering in this work, that is those which arise from an unmodified BF-type theory with the imposition of simplicity constraints (as the EPRL model), *do not admit a QpCH formulation.* Indeed, we have just proven that the history operator for these models cannot be unitary, so there is no notion of *unitary evolution* to be extracted from them. In hindsight, this could be expected: it is a well-known fact that the hamiltonian for general relativity vanishes identically, being composed purely of constraints. Recall from equation (2.68) that most spin-foam models are obtained as a sum-over-histories restricted by a delta function, which is supposed to impose those constraints. In this sense the path integral can be thought of as an implementation of a projector operator on the kinematical space of states, yielding the physical inner product. Schematically,

$$\int \mathrm{d}\lambda \int_{\psi,\phi} \mathcal{D}\varphi e^{i\int(\lambda C - \pi\dot{\varphi})\,\mathrm{d}\tau} = \langle\psi|\phi\rangle_{\text{phys.}} = \langle\psi|P|\phi\rangle_{\text{kin.}} = \int \mathcal{D}\varphi e^{-i\int \pi\dot{\varphi}\,\mathrm{d}\tau}\delta(C)\,. \tag{3.40}$$

where λ is a Lagrange multiplier imposing the constraint $C = 0$, φ represents the fields of the theory and π their conjugate momenta. The operator P acts as a projector onto the states satisfying the constraints. Since our history operator is in some sense induced by such a projector operator, its failure in satisfying unitarity can be justified.

Lastly we would like to note that, while we have ruled out a big class of models from being unitary, there exist in the literature certain constructions of spin-foam models which, unlike the ones studied here, propose amplitudes which do depend on the edge orientations of the dual complex [46, 47] (we note in passing that in [46] that such a dependence is implemented by restricting the domain of integration of the Lagrange multiplier (3.40) imposing the constraint). We expect the history operator of such "causal" models to be better behaved, and hypothesize that, under the requirement of unitarity of the operator, relevant constraints on the parameters of the theory (specially the fusion coefficients) could be obtained.

Summary and Outlook

We have explored in this work the spin-foam approach to quantum gravity, and argued towards its relevancy as a consistent, rigorous and physically well-motivated theory, furthermore implementing the main features of Einstein's general theory of relativity. The important conceptual aspects of these models, as the meaning of probability amplitudes and the role of spin-network states for spacetime, were also discussed. So too were the mathematical structures we consider to be fundamental in the formulation of these theories. We have also presented a succinct overview of the causal set program and related constructions, having in mind the objective of establishing a correspondence between spin-foam models and quantum causal histories.

Regarding the correspondence between the two frameworks, we have shown how a linear "history" operator can be constructed from spin-foam models, such that the models' amplitudes can be recovered as the usual transition amplitudes of the operator. We have moreover investigated the implications of requiring linearity on the evolution of quantum mechanical states inside causal loops, and derived from them a condition on the evolution operator characterizing the system. Using these obseervations we have formulated a prescription for establishing the correspondence between a spin-foam model and a quantum causal history, and we have shown that a large class of spin-foam models found in the literature do not admit such a correspondence, chiefly because the amplitudes of these models do not depend on the orientations of the spin-foam.

There remain still some open questions to be addressed in some later time. First of all, it would be interesting to apply our history operator construction to the causal spin-foam models found in the literature, as we reasonably expect that the condition of unitary could further constrain such models. Furthermore, the role of the Deutsch condition on spin-foams deserves to be investigated, eventually trough the formulation of a weaker version of the condition, as again we expect this condition to constrain the models. From the opposing point of view, the idea that spin-foam models do not necessarily need to satisfy a unitarity condition (which is demanded in quantum mechanics in a very different setting, where time is a parameter and not a dynamical physical entity) deserves to be considered and further studied.

Appendix A

Geometry of Gauge Theory

A.1 Connections on Principle Bundles

A great deal of the progress that has been made towards a quantum theory of gravity hinges on the reformulation of General Relativity in terms of objects that one would find in any gauge theory. In order to understand such a reformulation, and for completeness, we discuss here the mathematical setting of gauge theories, i.e. principal bundles, and how the usual physical fields one studies in such a class of theories arise out of geometric quantities on these bundles. Most of this section is taken from [48] and [49].

A.1.1 Principal Bundles

We start with the well-known definition of a principal bundle. A smooth bundle $P \xrightarrow{\pi} M$ with fiber G and local trivializations

$$\begin{aligned} \phi_i : \pi^{-1}(U_i) &\to U_i \times G \\ u &\mapsto (x, h) \end{aligned} \tag{A.1}$$

over an open cover $\{U_i\}$ of M is a *principal G-bundle* if it has G as a structure group acting on the left. Then there is a natural smooth fibre-preserving right action $P \triangleleft G$ of G on itself, given by $\phi_i(ug) = (x, hg)$. Note that this definition is independent of the trivialization, because under some other trivialization ϕ_j we have

$$\begin{aligned} \phi_i^{-1}(x, hg) &= \phi_j^{-1}(x, f_{ij}(hg)) \\ &= \phi_j^{-1}(x, (f_{ij}h)g) \\ &= ug \end{aligned}$$

for $f_{ij} \in G$. Notice moreover this action is both transitive (because the right action of G on itself is transitive) and free (because ϕ_i is a homeomorphism).

A useful construction to consider for principal bundles is the one of the *vertical tangent subspace* $V_uP = \ker \pi_{p*}$, intuitively understood as the space of vectors at $u \in P$ that are parallel to the fibers. One can then define $VP = \sqcup_{u\in P} V_u P$ in the obvious way as a sub-bundle of P. There is a very simple way in which such vectors can be constructed. Consider the Lie-algebra $\mathfrak{g}$ of G and the usual exponential map $\exp : \mathfrak{g} \to G$. To every element $X \in \mathfrak{g}$ we can associate a vector field $X^\#$, called the *fundamental vector field* on P associated to X: let $\gamma_u^X : t \mapsto u \exp(tX)$ be the curve starting at u generated by the right action, and

$$\begin{aligned} X_u^\# &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} u \exp tX \\ &= \gamma_u^X{}_* \left(\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \right) . \end{aligned}$$

It is easy to check that $X^\#$ is indeed vertical at every point u, and it turns out [49] that $\#_u : \mathfrak{g} \to V_u P$ is in fact a vector space isomorphism.

A.1.2 Connections

We may now define the notions of connection and curvature of a principal fiber bundle, and these objects will have a direct correspondence with the usual analogous notions for vector bundles.

As we saw in the previous subsection, there is a unique and well-defined notion of vertical vectors on the principle bundle, but there is no canonical choice of the complement, called the *horizontal tangent space* $H_u P$. A *connection* on a principle G bundle is then a unique choice of the smooth complement subspace $H_u P$ to $V_u P$ [49] such that

$$\begin{aligned} &\text{(i)} \quad T_u P = H_u P \oplus V_u P, \quad u \in P \\ &\text{(ii)} \quad H_{ug} P = R_{g*} H_u P, \quad g \in G \\ &\text{(ii)} \quad H_u P \ \text{ depends smoothly on } u \end{aligned}$$

(this definition of connection may seem too abstract and rather far from the usual intuition of using the connection to parallel transport vectors along a curve. As we shall see, however, transporting vectors in a unique manner along a curve is exactly what a choice of a horizontal space allows for).

It turns out that a systematic choice an horizontal space can be made by prescribing a *connection 1-form*, that is, a Lie-algebra-valued 1-form $\omega \in \Gamma(TP \otimes \mathfrak{g})$ satisfying

$$\begin{aligned} &\text{(i)} \quad \omega(X^\#) = X, \quad X \in \mathfrak{g} \\ &\text{(ii)} \quad R_g^* \omega = \mathrm{Ad}_{g^{-1}} \omega \,, \end{aligned}$$

and the horizontal subspace arises as its kernel

$$H_u P \coloneqq \{ X \in TP | \omega(X_u) = 0 \} \,. \tag{A.2}$$

One can check that the horizontal subspace defined in (A.2) satisfies all of the properties required in the first definition.

To define the curvature of the connection, we start by defining a differential operator acting on forms in P. Let $\phi \in \Omega^r(P)$ be such a form, and $X_1, ..., X_{r+1} \in T_uP$. The *exterior covariant derivative* D is a map $\mathrm{D} : \Omega^r(P) \to \Omega^{r+1}(P)$ defined by

$$\mathrm{D}\phi(X_1, ..., X_{r+1}) = \mathrm{d}\phi\left(\mathrm{hor}(X_1), , \mathrm{hor}(X_{r+1})\right) , \tag{A.3}$$

where d is the usual exterior derivative on the forms and hor denotes the horizontal component of the vector. The *curvature 2-form* Ω is then simply defined by the action of the exterior covariant derivative on the connection, i.e.

$$\Omega = \mathrm{D}\omega \in \Omega^2(P) \otimes \mathfrak{g} . \tag{A.4}$$

It turns out [49] (we omit the proof because we consider it to be more lengthy than useful) that equation (A.4) can be equivalently written as

$$\Omega = \mathrm{d}\omega + \omega \wedge \omega , \tag{A.5}$$

where the second term is to be understood as the wedge operation on the matrix components being multiplyied, and this equation is commonly called the *curvature structural equation*. The exterior covariant derivative also admits a simple formula for a particular family of forms in P, namely the forms α with values in some vector space V that are equivariant with respect to a represenation $\rho : G \to \mathrm{GL}(V)$, i.e.

$$R_g^* \alpha = \rho(g)^{-1} \alpha , \tag{A.6}$$

named *tensorial forms of type* ρ in the literature. It can be shown [50] that for this class of forms the exterior covariant derivative can be written as

$$\mathrm{D}\alpha = \mathrm{d}\alpha + \rho_*(\omega) \wedge \alpha , \tag{A.7}$$

where the wedge operator is to be understood as wedging each component in the matrix multiplication.

A.1.3 Local form of the connection

The connection on a principal bundle $P(G, M)$ can be easily pulled back to the base manifold, and we will see this determines a choice of a connection on vector bundles related to it. Let $\{U_i\}$ be an open covering of M and $s_i : M \to P$ local sections of P. Define the *local connection* on each open set U_i to be the pullback $A_i = s_i^* \omega$ of the connection. Note that it is exactly the local form of the connection $A_i \in \Omega(M) \otimes \mathfrak{g}$ that is used in physical gauge theories. One can show [48] that these local forms satisfy a compatibility condition in each $U_i \cap U_j \neq \emptyset$, with $s_j = s_i t_{ij}$, $t_{ij} \in G$, given by

$$A_j = t_{ij}^{-1} A_i t_{ij} + t_{ij}^{-1} \, \mathrm{d}t_{ij} . \tag{A.8}$$

Analogously, for two local sections s_1, s_2 over U with $A_1 = s_1^*\omega$, $A_2 = s_2^*\omega$, and $s_2(x) = s_1(x)g(x)$, the local field transforms as

$$A_2 = g^{-1}A_1 g + g^{-1}\,\mathrm{d}g \tag{A.9}$$

Regarding the curvature, the local form is similarly constructed with the pullback by a local section, $F_1 = s_1^*\Omega$, and from equation (A.5) one easily sees that it relates to the local curvature trough

$$F_1 = \mathrm{d}A_1 + A_1 \wedge A_1\,, \tag{A.10}$$

with a similar compatibility condition on $U_i \cap U_j \neq \emptyset$,

$$F_j = t_{ij}^{-1} F_i t_{ij}\,. \tag{A.11}$$

Finally, from equation (A.9), the right action of G, with $s_2 = s_1 g$, transforms the local curvature form by

$$F_2 = g^{-1}F_1 g\,. \tag{A.12}$$

A.1.4 Group of gauge transformations

Given a principal G-bundle $P \overset{\pi}{\to} M$ and a local section $s_1 : U \to P$ inducing a local connection form $A_1 = s_1^*\omega$, recall (A.9) that, under a smooth transformation of the section $s_2(x) = s_1(x)g(x)$, the induced local connection 1-form transforms as

$$A_2 = g^{-1}A_1 g + g^{-1}\,\mathrm{d}g\,, \tag{A.13}$$

with $A_2 = s_2^*\omega$.

It turns out that such gauge transformations can be seen as bundle isomorphisms of P that reduce to the identity on the base, i.e. maps of the form

$$\begin{aligned} f : P &\to P \\ \text{s.t.}\quad f(ua) &= f(u)a,\ a \in G\,. \end{aligned} \tag{A.14}$$

Since these maps are fiber-preserving, such a transformation may also be seen as a map

$$\begin{aligned} g : P &\to G \\ \text{s.t.}\quad g(ua) &= a^{-1}g(u)a,\ a \in G\,, \end{aligned} \tag{A.15}$$

and we identify $f(u) = ug(u)$. These can then be used to act on sections $s_2(x) = s_1(x)(g \circ s_1)(x)$ and induce gauge transformations of the connection as above. Notice that the set of g maps has a natural group structure which can be associated with the bundle isomorphisms f, and so one calls the group of such isomorphisms the *gauge group* $\mathcal{G}$ [11].

The gauge group also admits a nice geometrical description in terms of the associated bundle $P \times_{\mathrm{conj}} G$, where we identify $[u, g] = [uh, h^{-1}gh]$. Notice that there is a multiplication at the fibers $[u, g] \cdot [u, h] = [u, gh]$, so that the fibers are isomorphic to G. Consider now a section $\psi : \pi^{-1}(P) \to P \times G$. There is a unique $f \in \mathcal{G}$ s.t. $f(\psi^{(1)}(x)) = \psi^{(1)}(x) \cdot \psi^{(2)}(x)$. Moreover, for any $v \in P$, there exists a unique $h \in G$ s.t. $v = \psi^{(1)}(x) \cdot h$. It follows that

$$\begin{aligned} f(v) &= f(\psi^{(1)}(x) \cdot h) = f(\psi^{(1)}(x)) \cdot h \\ &= \psi^{(1)}(x) \cdot \psi^{(2)}(x) \cdot h = v \cdot h^{-1} \cdot \psi^{(2)}(x) \cdot h \,, \end{aligned}$$

and in particular $f(\psi^{(1)}) = \psi^{(1)} \cdot \psi^{(2)}$, from where we see the correspondence $\mathcal{G} \simeq \Gamma\left(P \times_{\text{conj}} G\right)$.

A.1.5 Parallel transport

A connection ω on a principal bundle $P \xrightarrow{\pi} M$ specifies a unique way of transporting points over it. It is realized by singling out a unique lift $\tilde{\gamma} : [0,1] \to P$ from a curve on the base $\gamma : [0,1] \to M$ such that the lift projects to the curve, *i.e.* $\pi \circ \tilde{\gamma} = \gamma$, and the tangent vectors to $\tilde{\gamma}$ in TP all lie in the horizontal subspace determined by ω.

One can construct a closed-form expression for the lift [48]. Given a local section $s : U \to P$, one writes $\tilde{\gamma}(t) = \big((s \cdot g) \circ \gamma\big)(t)$, with g given by

$$g\big(\gamma(t)\big) = \mathcal{P} \exp\left\{ - \int_{\gamma} s^{*}\omega \right\} \,, \tag{A.16}$$

where $\mathcal{P}$ denotes the path-ordering operation. This transport then specifies mappings $\Gamma_{\gamma} : P_{\gamma(0)} \to P_{\gamma(1)}$ satisfying $\Gamma(ug) = \Gamma(u)g$.

Finally, under a gauge transformation $\omega \to f^{*}\omega$, $f \in \mathcal{G}$, one can check that the transport transforms as

$$g(\gamma) \xrightarrow{f} f\big(\gamma(1)\big)^{-1} g(\gamma) f\big(\gamma(0)\big) \,. \tag{A.17}$$

A.1.6 Associated vector bundles

Given a principal bundle $P(M,G) \xrightarrow{\pi} M$, there is a way in which the connection there defined induces a connection on a related vector bundle. Consider the smooth left action of G on a k-dimensional vector space V through a representation $\rho : G \to \mathrm{Aut}(V)$, and define another action of G on the product bundle $P \times V$ in the following way:

$$\begin{gathered} G \times (P \times V) \to P \times V \\ (g, (u,v)) \mapsto (ug, \rho(g)^{-1} v) \end{gathered} \tag{A.18}$$

The quotient bundle $E \coloneqq (P \times V)/G$, also denoted $P \times_{\rho} V$, obtained through the identification $(u,v) \sim (ug, \rho(g)^{-1}v)$, together with the projection map $\pi_E(u,v) = \pi(u)$, is then called an *associated vector bundle* to the principal bundle.

An important result [50] regarding associated bundles, which we do not show here, is that there exists an isomorphism $I : \Gamma(\Lambda^k T^{*}M \otimes E) \to \Gamma_{\rho}(P \otimes V)$ between E-valued forms in M and V-valued forms in P that are equivariant with respect to ρ as in equation (A.6), i.e. tensor forms in P of the type ρ.

A very important feature of associated vector bundles to principal bundles is that a connection on the principal bundle fully determines a connection on the associated bundle. In fact [51], given a local section $s : U \subset M \to P$, and a local section $v : U \subset M \to V$, the induced connection on E acts locally on $e = [s, v]$ as

$$\nabla_X e = [s, \mathrm{d}v(X) + \rho_*(A(X))v] \tag{A.19}$$

where A is the local connection $A = s^*\omega$ induced by the connection 1-form ω of P. In particular, the connection acts on a frame $e_I = [s, \hat{e}_I]$ as

$$\nabla e_I = [s, \rho_*(A)^J{}_I \hat{e}_J]\,, \tag{A.20}$$

where $\hat{e}_I$ is an element of the canonical basis of V. By a small abuse of notation, this may also be written as $\nabla e_I = e_J \otimes \rho_*(A)^J{}_I$.

Another way to get to this expression for the induced connection is to consider first the induced exterior covariant derivative. As was discussed in the previous subsection, there exists an isomorphism I between tensorial forms in P of type ρ and E-valued forms on the base manifold. Having defined the exterior covariant derivative in P, acting on tensorial forms of type ρ as in (A.7), we can define an exterior covariant derivative for E-valued forms α in M simply by taking $\mathrm{D}\alpha = I^{-1}(\mathrm{D}I(\alpha))$. Doing so, we get [49] a map $\mathrm{D} : \Omega^k(M, E) \to \Omega^{k+1}(M, E)$ acting on $\alpha \otimes e_I$ as

$$\mathrm{D}(e_I \otimes \alpha) = e_I \otimes \mathrm{d}\alpha + \rho_*(A)^J{}_I e_J \wedge \alpha\,, \tag{A.21}$$

where $\alpha \in \Omega(M)$ and e_I is a canonical section of E. Note that D reduces to ∇ for 0-forms in M.

A.2 Frames for Vector Bundles

In this section we describe and discuss some standard results related to *frames* on vector bundles, *i.e.* ordered basis for the fibers of the bundle. The general treatment that follows will prove useful even when describing the tangent bundle of a smooth manifold, where a canonical choice of basis is available, which is the setting in which General Relativity is formulated. Most of this section follows the treatment by Chern in his book [52].

A.2.1 Bundles of Frames

We start by showing how a choice of a frame at every point for a vector bundle gives rise to a bundle of frames. Given a smooth vector bundle $E \xrightarrow{\pi} M$ of dimension $d+1$, there are homeomorphisms $\pi^{-1}(U_i) \to U_i \times \mathbb{R}^{d+1}$ associated to each open set U_i in an open cover $\{U_i\}$ of M, and hence we may construct an ordered set of $d+1$ linearly independent local sections $\{e_I\}$ of E. To such a set of sections we call a *frame* [52] of E over U_i. We can organize the frames of a vector bundle in the following way: consider the sets

$$\begin{gathered}\mathrm{Fr}(E_x) \coloneqq \big\{\{e_I\}_{I=0,\dots,d} \,|\, e \text{ is frame of } E_x = \pi^{-1}(x)\big\} \\ \mathrm{Fr}(E) \coloneqq \bigsqcup_{x \in M} \mathrm{Fr}(E_x)\end{gathered} \tag{A.22}$$

and a projection map $\rho : \mathrm{Fr}(E) \to M$. We endow the set $\mathrm{Fr}(E)$ with a topology by taking the trivializing maps ϕ_i given by

$$\begin{aligned} \phi_i : \rho^{-1}(U_i) &\to U_i \times GL(d+1, \mathbb{R}) \\ \big(x, \{e_I{}^\mu \partial_\mu\}\big) &\mapsto \big(x, e_I{}^\mu\big) \,, \end{aligned} \tag{A.23}$$

and demanding them to be homeomorphisms. In this way we obtain the so-called *frame bundle* $\mathrm{Fr}(E)$ associated to E, and a local smooth section of $\mathrm{Fr}(E)$ will then be a smooth choice of frames of E at each point in the neighborhood. Note that the transition functions are of the form

$$\begin{aligned} \phi_i \phi_j^{-1} : (U_i \cap U_j) \times GL &\to (U_i \cap U_j) \times GL \\ (x, e_I{}^\mu) &\mapsto (x, e_I{}^\mu \frac{\partial x'^\nu}{\partial x^\mu}) \,, \end{aligned} \tag{A.24}$$

and $\frac{\partial x'^\nu}{\partial x^\mu}$ is nonsingular (because x, x' are charts on the atlas of M), so the structure group of this bundle is precisely $GL(d+1, \mathbb{R})$. Trough the trivializing maps one sees that the fibers are homeomorphic to the structure group, and as such the frame bundle has naturally the structure of a GL principal bundle.

Given a frame bundle $\mathrm{Fr}(E)$, it is frequently useful to consider its dual bundle $\mathrm{Fr}(E)^*$. This is constructed the usual way, by defining its fibers to be the dual spaces of the fibers of $\mathrm{Fr}(E)$. The sections $\{\theta^I\}$ of the dual frame bundle are appropriately called *coframes*, and they satisfy the duality condition at each point:

$$\theta^I(e_J) = \delta^I{}_J \,. \tag{A.25}$$

While the GL frame bundle considered above describes any possible choice of frames in E, often one may be interested in restricting one's attention to a particular subset of those frames. This restriction is particularly important in physics, where the so-called *inertial frames* play a major role. To describe how other G frame bundles may arise as sub-bundles of the GL bundle, we focus our attention, as an example, in precisely those inertial frames. Let M be orientable, and equip the bundle $E \xrightarrow{\pi} M$ with a Lorentzian metric g. Then E has a natural global volume form vol [10] that can be used to specify the orientation of bases of fibers in E. Consider the subset of oriented and orthogonal frames at each point $x \in M$ relative to the Minkowski metric η,

$${}^\perp\mathrm{Fr}(E) = \{(x, e) \in \mathrm{Fr}(E) |\ \mathrm{vol}(\wedge_I^{d+1} e_I) > 0 \wedge g(e_I, e_J) = \eta_{IJ}\} \,, \tag{A.26}$$

and endow it with the subset topology and a projection map ${}^\perp\mathrm{Fr}(E) \xrightarrow{\xi} M$, with $\xi = \rho|_{{}^\perp\mathrm{Fr}(E)}$. The trivializing maps can be taken to be the same ones as in A.24 but with the restricted projection, and in this way ${}^\perp\mathrm{Fr}(E)$ becomes a sub-bundle of $\mathrm{Fr}(E)$ as the bundle of oriented orthogonal frames. Moreover, notice that the transition functions map an oriented orthogonal frame to another one, and such a mapping is given by definition by an $SO(d, 1)$ matrix. It follows that the structure group of this bundle is that Lie group, acting on the fibers as $e'_I{}^\mu \mapsto \Lambda_I{}^J e_J{}^\mu$, and ${}^\perp\mathrm{Fr}(E)$ can be identified with an $SO(d, 1)$ principal bundle. Again, a natural right action by $SO(d, 1)$ on the fibers, free and transitive, arises as above.

A.2.2 Frame technology

Most of the geometrical quantities one works with in General Relativity are more often than not described in terms of their components under the canonical basis for the tangent bundle coming from the atlas of the base manifold. Having defined frames and their bundles, we may now show how these objects can be described in terms of them, and this will prove very useful in problems related to quantum gravity. Most of this section follows Chern's excellent book [52].

Consider a set of smooth local sections $\{e_I\}$ over $U \subset M$ of the vector bundle $E \xrightarrow{\pi} M$, such that a general section can be written as $s = \alpha^I e_I$ for some $\alpha \in C^\infty(U)$. A connection ∇ on E is defined, as usual, as a map

$$\begin{aligned} &\nabla : \Gamma(E) \to \Gamma(T^*M \otimes E) \\ \text{s.t. (i)} \quad &\nabla(s_1 + s_2) = \nabla s_1 + \nabla s_2 \\ \text{(ii)} \quad &\nabla(\alpha s) = \mathrm{d}\alpha \otimes s + \alpha \nabla s \,, \end{aligned} \tag{A.27}$$

and $\nabla_X s = \nabla s(X)$, for some $X \in \Gamma(TM)$. Since the image of the connection is an element of $\Gamma(E \otimes T^*M)$, we may write locally

$$\nabla e_I = e_J \otimes \Gamma^J{}_{I\mu}\, \mathrm{d}x^\mu := e_J \otimes A^J{}_I \tag{A.28}$$

such that $A^J{}_I = \Gamma^J{}_{I\mu}\, \mathrm{d}x^\mu$ is a matrix of 1-forms in $\Omega(U)$, called the *connection matrix*.

Note that, in accordance with section A.1.3, the connection on E can be constructed from a principal connection 1-form ω on the frame bundle $\mathrm{Fr}(E)$. The frame bundle induces a connection on any associated bundle $\mathrm{Fr}(E) \times_\rho V$ through $\nabla[s, \hat{e}] = [s, \rho_*(A)\hat{e}]$, where $A = s^*\omega$, $\hat{e}$ is the canonical basis of V and s is a section of $\mathrm{Fr}(E)$. This connection can be transported to E through an isomorphism, so that one can think of the connection matrix as $A^J{}_I = (\rho_*(A))^J{}_I$. In the rest of this subsection we will, however, just assume that a connection on E is directly given.

Clearly, the choice of the forms $A^J{}_I$ depends on the particular frame one considers. Given any other frame $e'_J = e_I \Lambda^I{}_J$, we find (note that the exterior derivative acts on each component of Λ, seen as smooth functions over M)

$$\begin{aligned} \nabla e'_J &= e_I \otimes \mathrm{d}\Lambda^I{}_J + \nabla e_I \Lambda^I{}_J \\ &= e_I \otimes \mathrm{d}\Lambda^I{}_J + e_K \otimes A^K{}_I \Lambda^I{}_J \\ &= e'_L (\Lambda^{-1})^L{}_K \otimes (\mathrm{d}\Lambda^K{}_J + A^K{}_I \Lambda^I{}_J) \\ &= e'_L \otimes \left[(\Lambda^{-1})^L{}_K\, \mathrm{d}\Lambda^K{}_J + (\Lambda^{-1})^L{}_K A^K{}_I \Lambda^I{}_J \right] \,, \end{aligned} \tag{A.29}$$

so that, under a transformation of the frame, the forms transform as $A' = \Lambda^{-1}\, \mathrm{d}\Lambda + \Lambda^{-1} A \Lambda$, and we recover the result we obtained for the transformation of the principal connection A.9. From the exterior derivative of this transformation rule one finds $\Lambda\, \mathrm{d}A' - \mathrm{d}\Lambda \wedge A' = \mathrm{d}A\Lambda + A \wedge \mathrm{d}\Lambda$, and upon substituting $\mathrm{d}\Lambda = \Lambda A' + A\Lambda$ one gets the useful relation

$(\mathrm{d}A' + A' \wedge A') = \Lambda(\mathrm{d}A + A \wedge A)\Lambda^{-1}$. This nice transformation property (again, similar to the one of the principal curvature) warrants the definition of the object

$$\begin{aligned} F = \mathrm{d}A + A \wedge A \in \Gamma(\Lambda^2 T^* M \otimes E \otimes E^*) \\ F' = \Lambda^{-1} F \Lambda \,, \end{aligned} \tag{A.30}$$

to which we call the *curvature matrix* of the connection.

To understand the choice of nomenclature, consider $X, Y \in \Gamma(TU)$ and $s = \alpha^I e_I$ a section of E over U. Define the map

$$\begin{aligned} R(X,Y) : \Gamma(E) \to \Gamma(E) \\ s \mapsto e_J {F^J}_I(X,Y)\alpha^I \,, \end{aligned} \tag{A.31}$$

and note that

$$\begin{aligned} \nabla_X s &= e_K \left[\mathrm{d}\alpha^K(X) + {A^K}_I(X)\alpha^I\right] \\ &= e_K \left[X(\alpha^K) + {A^K}_I(X)\alpha^I\right] \end{aligned}$$

$$\begin{aligned} \nabla_Y \nabla_X s = e_K &\left[Y(X\alpha^K) + {A^K}_I(X) Y(\alpha^I) + Y({A^K}_I X)\alpha^I\right] \\ &+ e_L {A^L}_K(Y)\left[X(\alpha^K) + {A^K}_I(X)\alpha^I\right] \\ = e_K &\left[Y(X\alpha^K) + {A^K}_J(Y)X(\alpha^J) + {A^K}_I(X)Y(\alpha^I)\right. \\ &\left. + Y({A^K}_I X)\alpha^I + {A^K}_J(Y){A^J}_I(X)\alpha^I\right] \end{aligned}$$

$$\begin{aligned} \nabla_X \nabla_Y s - \nabla_Y \nabla_X s &= e_K \left\{[X,Y]\alpha^K + {A^K}_J(Y)X(\alpha^J) + {A^K}_I(X)Y(\alpha^I)\right. \\ &\quad \left. + \left({A^K}_J(X){A^J}_I(Y) - {A^K}_J(Y){A^J}_I(X)\right)\alpha^I\right\} \\ &= e_K \left\{[X,Y]\alpha^K + \left({A^K}_I([X,Y]) + \mathrm{d}{A^K}_I(X,Y)\right)\alpha^I\right. \\ &\quad \left. + \left({A^K}_J \wedge {A^J}_I(X,Y)\right)\alpha^I\right\} \\ &= e_K \left\{[X,Y]\alpha^K + {A^K}_I([X,Y])\alpha^I + {F^K}_I(X,Y)\alpha^I\right\} \\ &= \nabla_{[X,Y]} s + e_K {F^K}_I(X,Y)\alpha^I \,. \end{aligned}$$

We see that $R(X,Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}$, so $R(X,Y)$ is indeed the Riemman curvature tensor, and one has the relation

$$R(X,Y)e_I = e_J F(X,Y)^J{}_I \,. \tag{A.32}$$

In components, the tensor ${R^\alpha}_{\beta\mu\nu} = \mathrm{d}x^\alpha[R(\partial_\mu,\partial_\nu)\partial_\beta]$ can be related to the curvature matrix as

$$\begin{aligned} {R^\alpha}_{\beta\mu\nu} {e^I}_\alpha {e_J}^\beta &= \theta^I[R(\partial_\mu,\partial_\nu)e_J] \\ &= \theta^I F(\partial_\mu,\partial_\nu)^K{}_J e_K \\ &= F(\partial_\mu,\partial_\nu)^I{}_J \,, \end{aligned}$$

so that one may use the matrices ${e_I}^\mu$ to exchange internal indices and chart indices,

$$R^\alpha{}_{\beta\mu\nu} e^I{}_\alpha e^{J\beta} = F^{IJ}{}_{\mu\nu} \,, \tag{A.33}$$

and the Ricci scalar $R = g^{\beta\nu} \delta_\alpha{}^\mu R^\alpha{}_{\beta\mu\nu}$ is simply

$$R = F^{IJ}{}_{\mu\nu} \, {e_I}^\mu {e_J}^\nu \,. \tag{A.34}$$

Another useful geometric quantity one frequently need is the *torsion form*, which we define as a map $T : \Gamma(TU \otimes TU) \to \Gamma(E)$ with $T = T^I e_I$ and

$$T^I := \mathrm{d}\theta^I + A^I{}_J \wedge \theta^J \,. \tag{A.35}$$

Again, this object is indeed just the usual torsion, since, for $X, Y \in \Gamma(TU)$,

$$\begin{aligned} T^I(X,Y) e_I &= \left(\mathrm{d}\theta^I + A^I{}_J \wedge \theta^J\right)(X,Y) e_I \\ &= \left[X(\theta^I(Y)) - Y(\theta^I)(X) - \theta^I([X,Y]) + A^I{}_J(X)\theta^J(Y) + A^I{}_J(Y)\theta^J(X)\right] e_I \\ &= -[X,Y] + \nabla_X(\theta^I(Y)e_I) - \nabla_Y(\theta^I(X)e_I) \\ &= \nabla_X Y - \nabla_Y X - [X,Y] \,. \end{aligned}$$

With equation (A.21), the torsion form can simply be written as $T^I = \mathrm{D}\theta^I$.

To finish this section, we turn our attention to a very useful construction on smooth N-dimensional (semi-)Riemannian manifolds (M, g), called the *Hodge star operator*, mapping n-forms to $N - n$ forms, and defined by the relation $\alpha \wedge \star\beta = \langle \alpha, \beta \rangle \,\mathrm{vol}$, $\alpha, \beta \in \Omega^n(M)$. Explicitly,

$$\begin{aligned} \star : \Omega^n(M) &\to \Omega^{(N-n)}(M) \\ \star(\mathrm{d}x^{\mu_1} \wedge \ldots \mathrm{d}x^{\mu_n}) &= \frac{\sqrt{|g|}}{(N-n)!} \epsilon^{\mu_1 \ldots \mu_n}{}_{\mu_{n+1} \ldots \mu_N} \, \mathrm{d}x^{\mu_{n+1}} \wedge \ldots \mathrm{d}x^{\mu_N} \,. \end{aligned} \tag{A.36}$$

Since the frames are defined relative to the canonical basis in $\Omega(M)$, one can write the action of the Hodge star on a wedge product of coframes by referring to the above definition, and a nice closed-form expression is found as:

$$\begin{aligned} &\star\left(\theta^{I_1} \wedge \ldots \wedge \theta^{I_n}\right) = e^{I_1}{}_{\alpha_1} \ldots e^{I_n}{}_{\alpha_n} \frac{\sqrt{|\det g|}}{(N-n)!} \epsilon^{\alpha_1 \ldots \alpha_n}{}_{\alpha_{n+1} \ldots \alpha_N} \, \mathrm{d}x^{\alpha_{n+1}} \wedge \ldots \wedge \mathrm{d}x^{\alpha_N} \\ &= e^{I_1}{}_{\alpha_1} \ldots e^{I_n}{}_{\alpha_n} \frac{\det\theta}{(N-n)!} \epsilon^{\alpha_1 \ldots \alpha_n}{}_{\beta_1 \ldots \beta_{N-n}} e^{A_1}{}_{\alpha_{n+1}} {e_{A_1}}^{\beta_1} \ldots e^{A_{N-n}}{}_{\alpha_N} {e_{A_{N-n}}}^{\beta_{N-n}} \, \mathrm{d}x^{\alpha_{n+1}} \wedge \ldots \wedge \mathrm{d}x^{\alpha_N} \\ &= e^{I_1}{}_{\alpha_1} \ldots e^{I_n}{}_{\alpha_n} \frac{\det\theta}{(N-n)!} \epsilon^{\alpha_1 \ldots \alpha_n}{}_{\beta_1 \ldots \beta_{N-n}} {e_{A_1}}^{\beta_1} \ldots {e_{A_{N-n}}}^{\beta_{N-n}} \, \theta^{A_1} \wedge \ldots \wedge \theta^{A_{N-n}} \\ &= \frac{1}{(N-n)!} \, \epsilon^{I_1 \ldots I_n}{}_{A_1 \ldots A_{N-n}} \, \theta^{A_1} \wedge \ldots \wedge \theta^{A_{N-n}} \,, \end{aligned} \tag{A.37}$$

where $\det\theta = \frac{1}{N!} \epsilon_{I_1 \ldots I_N} \epsilon^{\mu_1 \ldots \mu_N} e^{I_1}{}_{\mu_1} \ldots e^{I_N}{}_{\mu_N}$ was used. From this it also follows that

$$\epsilon_{I_1 \ldots I_N} = \det\theta \, \epsilon_{\mu_1 \ldots \mu_N} {e_{I_1}}^{\mu_1} \ldots {e_{I_N}}^{\mu_N} \,. \tag{A.38}$$

Appendix B

Elements of Representation Theory on Compact Groups

One of the interesting features of the theories discussed in this work is that the geometrical character of classical gravity seems to reduce to combinatorial structures with group-theoretical data. In order to understand these structures, we collect here a couple of important results, based mostly on the very useful review presented in [53], focusing on the case of compact groups.

B.1 Basic terminology

We will mostly deal with unitary representations of groups. A *unitary representation* of G is a homomorphism

$$\begin{aligned} \rho : G &\to U(\mathcal{H}) \\ g &\mapsto \rho(g)\,, \end{aligned} \tag{B.1}$$

where $U(\mathcal{H})$ denotes the group of unitary operators on the Hilbert space $\mathcal{H}$. Moreover, $\mathcal{H}$ is called the *support* of ρ, and its dimension corresponds to the dimension of the representation. If $\mathcal{H}$ is of dimension $d < \infty$ then it is naturally isomorphic to $\mathbb{C}^d$, and $U(\mathcal{H})$ becomes the group of unitary matrices of order d.

Given two representations ρ, ρ' supported on $\mathcal{H}_\rho, \mathcal{H}_{\rho'}$ respectively, an *intertwiner* between them is a linear map

$$\begin{aligned} \iota : \mathcal{H}_\rho &\to \mathcal{H}_{\rho'} \\ \text{s.t.} \quad \iota(\rho(g)v) &= \rho'(g)\iota(v),\ \forall v, g \in \mathcal{H}_\rho, G\,. \end{aligned} \tag{B.2}$$

In other words, the intertwiner satisfies $\iota \circ \rho = \rho' \circ \iota$. Such maps are also called in the literature *G-linear* or *G-morphisms*. Note that, when the map is an automorphism, the intertwiner expresses a notion of invariance; it satisfies $\iota(\rho(g)v) = \rho(g)\iota(v)$. Furthermore, if there exist an intertwiner between two representations that is also an isomorphism, then the representations are deemed *equivalent*. A representation ρ on $\mathcal{H}_\rho$ is called *irreducible* if there are no subspaces of $\mathcal{H}_\rho$ under the action of ρ. Otherwise it is called, appropriately, *reducible*.

Given a unitary representation of finite dimension, we may define two useful functions on G. The *matrix elements* of ρ are, as usual, given by the map

$$\begin{aligned} \rho_{ij} &: G \to \mathbb{C} \\ g &\mapsto \langle e_i | \rho(g) e_j \rangle \ , \end{aligned} \tag{B.3}$$

where $\langle \cdot | \cdot \rangle$ denotes the euclidian inner product and e_i is an element of the canonical basis of the support of ρ. Moreover, the *character* χ_ρ of the representation ρ is given by the trace of the matrix associated to ρ,

$$\begin{aligned} \chi_\rho &: G \to \mathbb{C} \\ g &\mapsto \sum_i \rho_{ii}(g) \end{aligned} \tag{B.4}$$

Finally, a complex-valued function f which is invariant under conjugation, *i.e.* $f(hgh^{-1}) = f(g)$, is called a *class function.* Since the trace of a matrix is invariant under cyclic permutations of its arguments, it follows that the character is a class function.

B.2 The Haar measure and harmonic analysis

On every locally compact group G there exists a natural choice of measure, the *Haar measure* $\mathrm{d}g$, which is either invariant under left translations or right translations (its constructed will not be discussed here, where we have in mind only its applications). For compact groups in particular, the Haar measure can be shown to be bi-invariant. The (normalized) Haar integral is then the linear functional

$$\begin{aligned} I &: C_c(G) \to \mathbb{C} \\ I(f) &= \int_G \mathrm{d}g\, f(g)\,, \end{aligned} \tag{B.5}$$

satisfying $\int_G \mathrm{d}g = 1$. With it one can define the space $L^2(G)$ of square-integrable functions on G, with inner product[1]

$$\langle f|g \rangle = \int_G \mathrm{d}g\, f^*(g) h(g)\,. \tag{B.6}$$

Having a bi-invariant measure on compact groups allows us to construct a theory of harmonical analysis on them, which turns out to be an immensely powerful tool. To see how this is done, we first need to discuss *Schur's orthogonality relations*:

[1] we use throughout the notation $\cdot^*$ to denote complex-conjugation. Please take care that frequently in the literature one uses $\rho^*(g) = \rho(g^{-1})^T$ to denote the dual representation, although here we mean only complex conjugation. In any case, for unitary representations the conjugate and the dual coincide.

Theorem B.2.1 (Schur's orthogonality relations)**.** Given two unitary irreducible representations ρ, ρ' of a compact group G, their matrix elements satisfy

$$\langle \rho_{ij} | \rho'_{kl} \rangle = \begin{cases} \frac{1}{\dim(\rho)} \delta_{ik}\delta_{jl} & \text{if } \rho \sim \rho' \\ 0 & \text{otherwise} \end{cases} \tag{B.7}$$

As a an immediate corolary, their characters satisfy the orthogonality relations

$$\langle \chi_\rho | \chi_{\rho'} \rangle = \begin{cases} 1 & \text{if } \rho \sim \rho' \\ 0 & \text{otherwise}\,. \end{cases} \tag{B.8}$$

Another extremely useful tool that we will need is the *Peter-Weyl* theorem. Here we collect a couple of important facts:

Theorem B.2.2 (F.Peter-H.Weyl)**.** Let G be a compact group. Then it holds that:

- an orthonormal basis for the space $L^2(G)$ is given by the set
$$\left\{ \sqrt{\dim(\rho^\lambda)}\, \rho^\lambda_{ij} \,|\, \lambda \in \Lambda \right\}\,, \tag{B.9}$$
where Λ denotes the set of equivalence classes of unitary irreducible representations of G, and each class is labeled by λ.
- the set $\left\{ \chi_{\rho^\lambda} \,|\, \lambda \in \Lambda \right\}$ is an orthonormal basis of the class functions in $L^2(G)$.
- the dimension of ρ equals the image of e under χ_ρ.
- a finite dimensional unitary representation ρ of G is irreducible iff $\langle \chi_\rho | \chi_\rho \rangle = 1$.
- two unitary irreducible representations are equivalent iff their characters coincide.

Now we are ready for the harmonic expansion. Let G be again a compact group and $f \in L^2(G)$. Then f may be expanded as

$$f = \sum_{\lambda \in \Lambda} \dim(\rho^\lambda) \sum_{ij} \hat{f}^\lambda_{ij} \rho^\lambda_{ij} \tag{B.10}$$

where $\hat{f}^\lambda_{ij}$ are the Fourier harmonics with respect to the Peter-Weyl basis, *i.e.*

$$\hat{f}^\lambda_{ij} = \langle \rho^\lambda_{ij} | f \rangle = \int_G \mathrm{d}g\, f(g) \rho^\lambda_{ij}(g)^*\,. \tag{B.11}$$

In the special case of class functions, it is easy to show that the expansion reduces to

$$f = \sum_{\lambda \in \Lambda} \hat{f}^\lambda \chi_{\rho^\lambda} \tag{B.12}$$

where $\hat{f}^\lambda = \langle \chi_{\rho^\lambda} | f \rangle$. Finally, an analagous version of the Parseval identity also holds in this case, and it is given by the equality

$$||f||^2 = \int_G \mathrm{d}g\, |f(g)|^2 = \sum_{\lambda \in \Lambda} \sum_{ij} |\hat{f}^\lambda_{ij}|^2 \tag{B.13}$$

B.2.1 The bi-regular representation on $L^2(G)$

The theorem (B.2.2) referenced in the previous subsection states that the matrix elements of the irreducible unitary representations of G span the space os square-integrable functions $L^2(G)$. Here we rewrite this statement in a manner that will be useful to define the states of the theories discussed in our context.

First, note that for each $\lambda \in \Lambda$ one may consider the Hilbert spaces $\mathcal{V}_i^\lambda$, $i = 1, ..., \dim(\rho^\lambda)$ spanned by the rows of the matrices associated to λ, *i.e.*

$$\mathcal{V}_i^\lambda = \mathrm{span}_j\{\rho_{ij}^\lambda\}\,. \tag{B.14}$$

It is clear from the defining property of representations $\rho(gh) = \rho(g)\rho(h)$ that these subspaces are invariant under the *right-regular* unitary representation R acting on $L^2(G)$ via

$$\begin{gathered} R(h)f(g) = f(gh), \quad f \in L^2(G) \\ R(h)\sum_j c_j\rho^\lambda_{(i)j}(g) = \sum_j c_j\rho^\lambda_{(i)j}(gh) = \sum_{j,k} c_j\rho^\lambda_{(i)k}(g)\rho^\lambda_{kj}(h) \\ R(h) : \mathcal{V}_i^\lambda \to \mathcal{V}_i^\lambda \\ v \mapsto \rho^\lambda(h)v\,. \end{gathered} \tag{B.15}$$

It follows that the representation

$$\begin{gathered} R \otimes R^* : G \times G \to U(\mathcal{V}_i^\lambda \otimes (\mathcal{V}_i^\lambda)^*) \\ (g_1, g_2) \mapsto \rho^\lambda(g_1) \otimes \rho^{*\lambda}(g_2) \end{gathered} \tag{B.16}$$

is an irreducible representation on $\mathcal{V}_i^\lambda \otimes (\mathcal{V}_i^\lambda)^*$. Since, from the Peter-Weyl theorem, we know that $L^2(G)$ is spanned by the matrix elements of the unitary irreducible representations ρ_{ij}^λ, and using the fact that finite vector spaces of the same dimension are naturally isomorphic, we may decompose $L^2(G)$ as

$$\begin{gathered} \bigoplus_{\lambda\in\Lambda} \mathcal{V}^\lambda \otimes (\mathcal{V}^\lambda)^* \to L^2(G) \\ (v, \alpha) \mapsto f(\cdot) = \alpha(\rho(\cdot)v) \end{gathered} \tag{B.17}$$

where $\mathcal{V}^\lambda$ is the space spanned by any of the rows of ρ^λ. The representation $R \otimes R^*$ acts on this space by

$$f(g) = \alpha(\rho(g)v) \mapsto \rho^{*\lambda}(g_2)\alpha(\rho(g)\rho(g_1)v) = f(g_2^{-1}gg_1)\,, \tag{B.18}$$

so the representation we have constructed is equivalent to the *bi-regular* representation $\tau(g_1, g_2)f(g) = f(g_2^{-1}gg_1)$.

To summarize, our result is that the bi-regular representation τ on $L^2(G)$ can be decomposed in irreducible representations as

$$\begin{gathered} L^2(G) \simeq \bigoplus_{\lambda\in\Lambda} \mathcal{V}^\lambda \otimes (\mathcal{V}^\lambda)^* \\ \tau \simeq \bigoplus_{\lambda\in\Lambda} \rho^\lambda \otimes \rho^{*\lambda}\,. \end{gathered} \tag{B.19}$$

B.3 Intertwiners and invariant elements

The intertwining linear maps discussed above play an important role in the description of the space of invariant elements of the support of a representation. First, consider the following object

$$\begin{aligned} \pi : \mathcal{H} &\to \mathrm{Inv}_\rho(\mathcal{H}) \\ v &\mapsto \left(\int_G \mathrm{d}g\, \rho(g) \right) v \end{aligned} \tag{B.20}$$

where $\mathrm{Inv}_\rho(\mathcal{H}) = \{v \in \mathcal{H} \,|\, \rho(g)v = v,\ \forall g \in G\}$ denotes the set of invariant elements of ρ. Clearly the image of π is in $\mathrm{Inv}_\rho(\mathcal{H})$ because the measure is bi-invariant; indeed, we have $\rho(h) \left(\int_G \mathrm{d}g\, \rho(g) \right) v = \left(\int_G \mathrm{d}g\, \rho(hg) \right) v = \left(\int_G \mathrm{d}g\, \rho(g) \right) v$. Moreover, the map π is a projector, since for the exact same reason it holds that $\pi^2 = \pi$.

Our goal now is to establish a correspondence between the space of intertwiners and the space of invariant elements. To this end, consider two unitary representations ρ, ρ' supported in $\mathcal{H}_\rho, \mathcal{H}_{\rho'}$, respectively. We can construct a unitary representation supported in $\mathrm{Hom}(\mathcal{H}_\rho, \mathcal{H}_{\rho'})$ as

$$\begin{aligned} \theta : G \times G &\to U(\mathrm{Hom}(\mathcal{H}_\rho, \mathcal{H}_{\rho'})) \\ (g_1, g_2) &\mapsto \rho'(g_1) \circ \cdot \circ \rho(g_2^{-1})\,, \end{aligned} \tag{B.21}$$

so that $A \in \mathrm{Hom}(\mathcal{H}_\rho, \mathcal{H}_{\rho'})$ is mapped to $\rho'(g_1) \circ A \circ \rho(g_2^{-1})$. In other words, θ is given by the action of $\rho'(g_1)\rho^*(g_2)$. Now we have the following theorem.

Theorem B.3.1. There is a correspondence

$$\mathrm{Inv}_\theta(\mathrm{Hom}(\mathcal{H}_\rho, \mathcal{H}_{\rho'})) \simeq \mathrm{Int}(\mathcal{H}_\rho, \mathcal{H}_{\rho'})\,, \tag{B.22}$$

where $\mathrm{Int}(\mathcal{H}_\rho, \mathcal{H}_{\rho'})$ denotes the space spanned by the intertwiners $\mathcal{H}_\rho \to \mathcal{H}_{\rho'}$.

Proof:
Consider the space $\mathcal{H} = \mathcal{H}_\rho \otimes \mathcal{H}^*_{\rho'}$. We have a linear bijection

$$\begin{aligned} \mathrm{Int}(\mathcal{H}, \mathbb{C}) &\to \mathcal{H} \\ \iota &\mapsto |\iota\rangle \quad \text{s.t.}\ \ \iota(v) = \langle v|\iota\rangle\,, \end{aligned} \tag{B.23}$$

and clearly $\mathrm{Int}(\mathcal{H}, \mathbb{C}) \simeq \mathrm{Int}(\mathcal{H}_\rho, \mathcal{H}_{\rho'})$. From the definition of intertwiner $\iota(\theta(g_1, g_2)v) = \iota(v)$, so it follows that $\iota(\theta(g_1, g_2)v) = \langle \theta^\dagger(g_1, g_2)v|\iota\rangle = \langle v|\theta(g_1, g_2)\iota\rangle = \langle v|\iota\rangle\,, \forall v \in \mathcal{H}$, so indeed $\theta(g)\,|\iota\rangle = |\iota\rangle$. Intertwiners are thus identified with invariant elements. The converse identification is analogous. ■

As an immediate corollary, from the natural correspondence $\mathrm{Hom}(\mathcal{H}_\rho, \mathcal{H}_{\rho'}) \simeq \mathcal{H}_\rho \otimes \mathcal{H}^*_{\rho'}$, we have the important result

$$\mathrm{Inv}_\theta(\mathcal{H}_\rho \otimes \mathcal{H}^*_{\rho'}) \simeq \mathrm{Int}(\mathcal{H}_\rho, \mathcal{H}_{\rho'})\,. \tag{B.24}$$

Appendix C

Diagrammatics of Invariants

Around 1971 Roger Penrose proposed a diagrammatic notation for tensor operators [54]. Some years later this notation was flushed out and used extensively for computations and proofs in the context of group and representation theory. This notation turns out to be very useful for our purposes, so we review it here, following mainly [55].

C.1 Basic notation

We will be interested in finite-dimensional complex vector spaces and their duals, denoted V and V^*, respectively. A *tensor* is given by the matrix elements $x^{a_1...a_m}_{b_1...b_n}$ of some $x \in V^m \otimes V^{*n}$, such that the upper indices are associated to V and the lower ones to V^*.

Now, suppose that the vector space V under consideration is the support of some unitary representation (ρ, V) of a group G. According to the discussion in Section B.3, one may consider the set of invariant elements of V, *i.e.*

$$\mathrm{Inv}_\rho(V) = \{v \in V \mid \rho(g)v = v,\ \forall g \in G\}\,, \tag{C.1}$$

and for some tensor $x^{a_1...a_m}_{b_1...b_n}$ one can write in index notation

$$x^{c_1...c_m}_{d_1...d_n} = \rho(g)^{c_1}_{a_1}...\rho(g)^{c_m}_{a_m}\rho^*(g)^{b_1}_{d_1}...\rho^*(g)^{b_n}_{d_n}\ x^{a_1...a_m}_{b_1...b_n}\,. \tag{C.2}$$

In the section mentioned above we showed that one can construct elements of $\mathrm{Inv}_\rho(V)$ using the projector of equation (B.20). In particular, the invariant elements of the product $V^m \otimes V^{*n}$ will be the images of the projector

$$\pi_{V^m\otimes V^{*n}} = \int_G \mathrm{d}g \bigotimes_{i=1}^{m} \rho(g) \bigotimes_{j=1}^{n} \rho^*(g)\,. \tag{C.3}$$

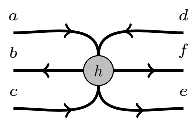
We will now construct a diagrammatic language for invariant tensors of some unitary representation (ρ, V). An invariant tensor is represented by a labeled vertex and a set of oriented edges (eventually labeled by the indices of the tensor). The orientation and ordering of the external edges conforms to the following two rules:

1. incoming edges are associated to upper indices, while outgoing edges are associated to lower indices;

2. the ordering of the indices on the tensor induces a labeling of the indices on the edges in a counter-clockwise direction.

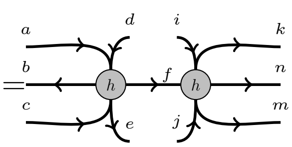
As an example of these rules, the invariant tensor $h \in V^m \otimes V^{*n}$ would be represented by the diagram[1]

$$h^{acd}_{bef} = \tag{C.4}$$



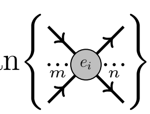
Moreover, as one would intuitively expect from the diagrammatic representation, a contraction of two tensors is represented by joining the appropriate edges of those two tensors. Referring to our previous example, the contraction of h with itself would be represented by

$$h^{acd}_{bef} h^{ifj}_{mnk} = \tag{C.5}$$



Now we consider some basis of the space of invariants $\mathrm{Inv}_{\rho\otimes\rho^*}(V^m \otimes V^{*n})$ given by d linearly independent invariant tensors, where d is the dimension of the space. Such a basis could be represented by

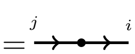
$$\mathrm{Inv}_{\rho\otimes\rho^*}(V^m \otimes V^{*n}) = \mathrm{span} \left\{ \cdots \right\}_{i=1,\ldots,d} , \tag{C.6}$$



where each invariant tensor has m incoming legs and n outgoing legs. Interestingly, sometimes one may be able to express some of the elements of this basis in terms of products of lower-rank tensors in the basis of a lower dimensional tensor product of the vector spaces. One calls a tensor of the space of invariants *primitive* if such a decomposition is however not possible.

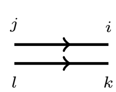
Primitive tensors will be represented with simple vertices. For example, the Kronecker delta is always one such primitive invariant tensor, and we represent it as

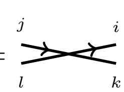
$$\delta^j_i = \, , \tag{C.7}$$



such that higher ordered tensors could be constructed, for example, by drawing two lines together

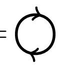
$$\delta^j_i \delta^l_k = \qquad \delta^j_k \delta^l_i = \, . \tag{C.8}$$



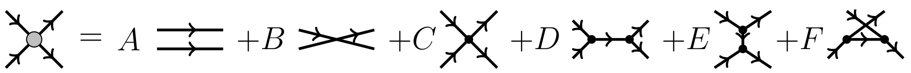


[1] We would like to acknowledge that most of the diagrams in this work were drawn using the TikZiT package [56].

By the way, it follows from the rule of contracting indices by joining lines that a diagram with no open ends represents a trace, and for the particular case of the delta diagram that trace corresponds to the dimension of the vector space,

$$\dim(V) =$$  . (C.9)

With the notion of a primitive in place, we can now rewrite our basis for the space of invariants. Given all the diagrams of equation (C.6), some of those diagrams will be primitive and some will be constructed from other primitive diagrams. As an example, for the case of $\mathrm{Inv}_{\rho\otimes\rho^*}(V^2 \otimes V^{*2})$, this means that any invariant tensor can be expressed as



(C.10)

with an eventual labeling of each primitive diagram if there is more than one primitive of the same rank. This summarizes the basic notation.

C.1.1 Clebsch-Gordan coefficients

Suppose that one is given a group G and two unitary representations μ, ν supported on V_μ, V_ν so that their tensor product can be factorized into a direct sum of irreducible unitary representations, *i.e.* $V_\mu \otimes V_\nu \simeq \bigoplus_\lambda V_\lambda$. Choose some orthonormal basis of $\bigoplus_\lambda V_\lambda$ arranged in a unitary matrix C. Consider moreover the projector $P_\lambda = \mathbf{0} \oplus ... \oplus \mathrm{id}_{V_\lambda} \oplus ... \oplus \mathbf{0}$ into the λ-th vector space. The *Clebsch-Gordan coefficients* (which we will call *clebsches* following [55]), are the linear maps

$$\begin{aligned} C_\lambda^{\mu\nu} &: V_\mu \otimes V_\nu \to V_\lambda \\ v &\mapsto \star(P_\lambda \circ C)v\,, \end{aligned} \tag{C.11}$$

which express the projection of an element of $V_\mu \otimes V_\nu$ to the space V_λ in terms of the basis of this space. The notation $\star(P_\lambda \circ C)$ is meant to represent that one takes the single $[\dim(V_\mu \otimes V_\nu) \times \dim(V_\lambda)]$ non-zero sub-matrix of $P_\lambda \circ C$. There is also a conjugate mapping

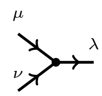
$$\begin{aligned} C_{\mu\nu}^\lambda &: V_\lambda \to V_\mu \otimes V_\nu \\ u &\mapsto \star(C^\dagger \circ P_\lambda)u\,, \end{aligned} \tag{C.12}$$

such that $C_{\mu\nu}^\lambda = \left(C_\lambda^{\mu\nu}\right)^\dagger$. Notice that, directly from the definition of the clebsches, we have the identities

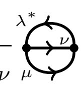
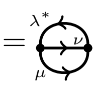
$$\begin{aligned} C_\lambda \circ C^\lambda &= P_\lambda \\ C^\rho \circ C_\lambda &= \delta_\lambda^\rho \, \mathrm{id}_{V_\lambda} \\ \sum_\lambda C_\lambda \circ C^\lambda &= \mathrm{id}_{V_\mu \otimes V_\nu} \end{aligned}\,, \tag{C.13}$$

making explicit the construction of the projector and the completeness relation (we omitted the $\mu\nu$ labels). Moreover, each clebsch is, according to its definition, an intertwiner.

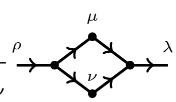
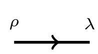
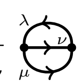
We may now consider the tensor components associated to the clebsches. The clebsch $C_{\lambda}^{\mu\nu} : V_{\mu} \otimes V_{\nu} \to V_{\lambda}$ canonically induces a map $V_{\mu} \otimes V_{\nu} \otimes V_{\lambda}^{*} \to \mathbb{C}$ giving its components. Therefore, we may also think of this tensor as a *bona fide* clebsch from the tensor product $V_{\mu} \otimes V_{\nu} \otimes V_{\lambda}^{*}$ representation to the scalar one, which we represent by $C_{0}^{\mu\nu\lambda^{*}}$. These clebsches are then invariant tensors, and as such they then deserve a diagrammatic expression of their own:

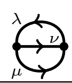
$$C_{0}^{\mu\nu\lambda^{*}} = \frac{1}{\sqrt{a_{\mu\nu\lambda}}}$$

, (C.14)

where we labeled the edges with the representations (rather than indices) and $a_{\lambda\mu\nu}$ is a normalization factor that is required for the the clebsches to be normalized. It is found using the second line of equation (C.13),

$$C_{\mu\nu\lambda^{*}}^{0} \circ C_{0}^{\mu\nu\lambda^{*}} = \frac{1}{a_{\lambda^{*}\mu\nu}}$$

$\Rightarrow a_{\lambda^{*}\mu\nu} =$

. (C.15)

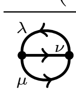
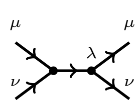
Moreover, since we can relate $C_{0}^{\mu\nu\lambda^{*}}$ to the tensor of $C_{\lambda}^{\mu\nu}$, we ought to be able to write this last clebsch using the same diagram. We can write $C_{\lambda}^{\mu\nu} = \frac{a_{\lambda\mu\nu}}{b_{\lambda\mu\nu}}$, and determine the factor $b_{\lambda\mu\nu}$ through

$$\frac{1}{b_{\lambda\mu\nu}}$$

$\overset{!}{=}$

$\Rightarrow \frac{1}{b_{\lambda\mu\nu}}$

$\overset{!}{=} \dim(\lambda)$ (C.16)

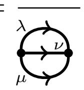
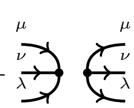
$\Rightarrow b_{\lambda\mu\nu} =$

$/ \dim(\lambda)$

Notice that, since the scalar representation has dimension one, we may use the factor $b_{\lambda\mu\nu}$ in full generality to represent any clebsch.

Now we may construct the projectors using the 3-vertex. According to (C.13), the projector $P_{\lambda} : V_{\mu} \otimes V_{\nu} \to V_{\lambda}$ into the λ-subspace is given by

$P_{\lambda} = \dim(\lambda) /$


, (C.17)

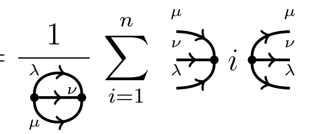
while the projector $P_{0} : V_{\mu} \otimes V_{\nu} \otimes V_{\lambda} \to \mathbb{C}$ is represented by

$P_{0} = 1 /$


. (C.18)

In these two equations one can see the usefulness of this pictorial representation: not only we were able to construct the projectors using different combinatorics of the same 3-vertex symbol, diagrams are also very suggestive of how the mapping is made.

Finally, we may represent the projector acting on unitary representations $\pi : V_\mu \otimes V_\nu \otimes V_\lambda \to \mathrm{Inv}_{\mu\otimes\nu\otimes\rho}(V_\mu \otimes V_\nu \otimes V_\lambda)$ of equation (B.20). It is given by

$$\pi = \int_G dg\, \mu(g) \otimes \nu(g) \otimes \lambda(g) =$$



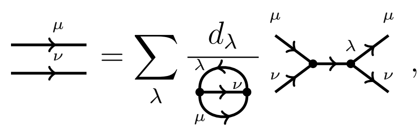
(C.19)

where, by summing over i, we are accommodating for the possibility that the tensor product of the representations decomposes into more than one irreducible representation of dimension zero.

C.1.2 Recouplings

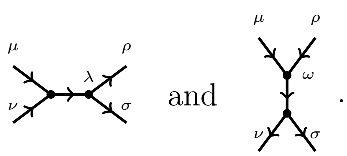
We end this short review with a few identities using in the reduction of diagrams to simpler ones, also called *recoupling*.

First, notice that the orthogonality of the clebsches in equation (C.13) can be written diagrammatically as



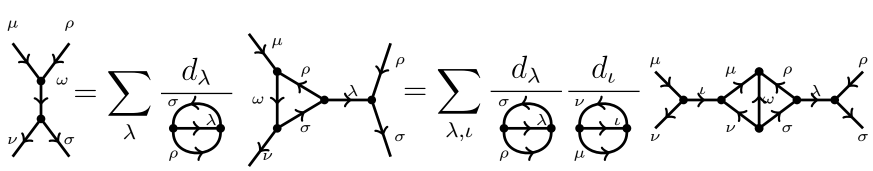
(C.20)

and we can use this identity to figure out a relationship between the projectors



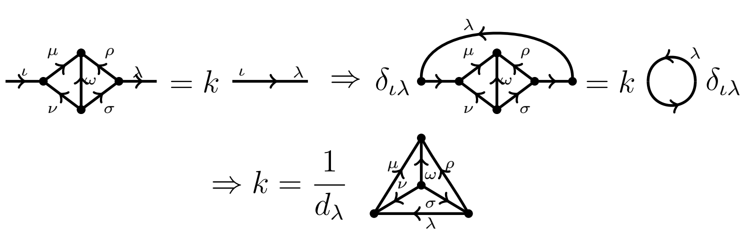
(C.21)

Starting from the diagram on the right, we can substitute twice, on the left and on the right, the orthogonality relation (C.20), and find



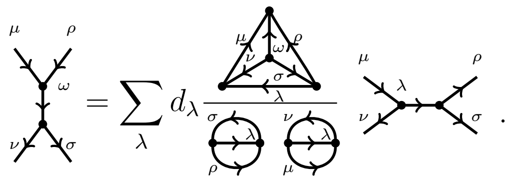
(C.22)

We can reduce this last diagram to a contraction of two clebsches times a factor, which we can find by fully contracting the diagram



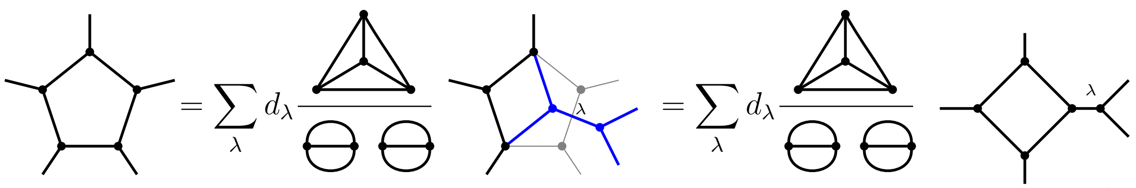
(C.23)

so our final formula for the recoupling is



(C.24)

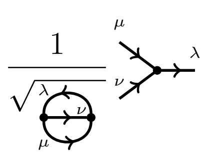
The usefulness of this recoupling formula is that it allows us to reduce any diagram with loops to a tree diagram, by substituting an edge with two vertices with a vertex with two edges. As an example, the equation below shows the reduction of a loop with five vertices to a loop with four



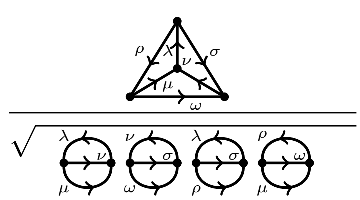
(C.25)

In this way, any closed diagram can be re-summed into a series of *teta* and *tetrahedra* bubble graphs, so knowing these particular numbers for some group representation is enough to compute any trace of a product of invariants. In physic one frequently used the *Wigner* $3j$ and $6j$ symbols for this purpose. They are related to our bubble diagrams through

$$\begin{pmatrix} \lambda & \mu & \nu \\ a & b & c \end{pmatrix} =$$



$$\begin{Bmatrix} \lambda & \mu & \nu \\ \omega & \sigma & \rho \end{Bmatrix} =$$



. (C.26)

Note that on the $3j$ symbol the Latin letters denote components, while the Greek ones denote the representation.

Appendix D

Identities in SL(2,ℂ)

This chapter is supposed to be a very superficial overview of important results from the representation theory of the double covering of the Lorentz group. The perspective is simply one of collecting tools, without any detail on their proper construction. The content of this chapter was collected in its entirety from [57, 58].

D.1 Representations of $L^2(SL(2,\mathbb{C}))$

D.1.1 The space of homogeneous functions

We start by considering the space $D_{(n_1,n_2)}$ of all homogeneous functions of degree $(n_1 - 1, n_2 - 1) \in \mathbb{C}^2$ that are smooth in the punctured space $\mathbb{C}^{2*}$. We denote $\chi = (n_1, n_2)$ for simplicity. Homogeneity requires for $f \in D_\chi$ that

$$f(e^{i\theta}x, e^{i\theta}y) = e^{i\theta(n_1-n_2)}f(x,y)\,, \tag{D.1}$$

so in order for the functions to be well defined we must demand $n_1 - n_2 \in \mathbb{Z}$. There exists a suitable topology in which D_χ becomes a topological space. An action of $SL(2,\mathbb{C})$ on D_χ can now be written as

$$\rho^\chi(g)f(x,y) = f(g^T(x,y))\,, \tag{D.2}$$

where we think of the argument of f as an horizontal complex vector. It turns out that this action constitutes a continuous representation of $SL(2,\mathbb{C})$ on D_χ.

An alternative realization of $D_{(n_1,n_2)}$ can be given in terms of functions on the complex 3-sphere $S^3 \subset \mathbb{C}^2$. Indeed, for any point (ω_1, ω_2) on the sphere and any another point $(r\omega_1, r\omega_2), r \in \mathbb{R}^*$ in $\mathbb{C}^{2*}$ we find

$$f(r\omega_1, r\omega_2) = r^{n_1+n_2-2}f(\omega_1,\omega_2)\,, \tag{D.3}$$

so to each element of D_χ there corresponds a smooth function on the sphere. Conversely, to every smooth function on the sphere satisfying $f(e^{i\theta}x, e^{i\theta}y) = e^{i\theta(n_1-n_2)}f(x,y)$, called the *covariance property*, there corresponds an element of D_χ. Notice further that there exists a diffeomorphism

$$\begin{aligned} d : SU(2) &\to S^3 \in \mathbb{C}^2 \\ u &\mapsto (u_{21}, u_{22}) \,, \end{aligned} \tag{D.4}$$

so a very useful realization of D_χ is given by the $SU(2)$ functions $\phi = f \circ d$ satisfying the covariance condition

$$\phi(\gamma u) = e^{i\theta(n_1 - n_2)}\phi(u) \,, \quad \gamma = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} . \tag{D.5}$$

D.1.2 Unitarity of representations

It turns out that there are only two cases in which one can associate an hermitian positive-definite functional $\langle\cdot|\cdot\rangle$ to D_χ that satisfies the invariance property

$$\langle \rho^\chi \psi | \rho^\chi \phi \rangle = \langle \psi | \phi \rangle \ . \tag{D.6}$$

The relevant cases are

1. When $n_1 = \frac{1}{2}(n + ip)$ and $n_2 = \frac{1}{2}(-n + ip)$, for $n \in \mathbb{Z}, p \in \mathbb{R}$, and the functional is given by

$$\langle \psi | \phi \rangle = \int_{SU(2)} \mathrm{d}u \ \psi^*(u)\phi(u) \,. \tag{D.7}$$

   These particular representations are said to be in the *principal series.*

2. When $n_1 = n_2 = p \in \mathbb{R} \setminus \{0\}$ for $-1 < p < 1$. These representations are said to be in the *complementary series.*

Using these functionals one may construct the norm $||\cdot|| = \langle\cdot|\cdot\rangle$ and complete D_χ in that norm, making D_χ into a Hilbert space H_χ. The operators ρ^χ can be uniquely extended to that Hilbert space, and in this way the principal and complementary series define unitary representations of $SL(2,\mathbb{C})$ on H_χ. It furthermore turns out that there is an equivalence of representations in the principal series for $D_\chi = D_{-\chi}$.

D.1.3 The canonical basis

By the Peter-Weyl theorem of subsection B.2.2, we know that the matrices ρ^λ_{ij} of the irreducible unitary representations $\lambda \in \Lambda$ of a compact group G form a complete orthonormal basis for $L^2(G)$. In the case of $SU(2)$ these are the Wigner $D^j_{m_1,m_2}$ matrices, where j is the usual spin labeling the representation and $|m_{1,2}| \leq j$. For the case at hand, one can show that the covariance condition of functions in H_χ induces a restriction of this basis. The restricted basis then becomes an orthonormal basis for the space $H_\chi \subset L^2(SU(2))$ in terms of its $SU(2)$ realization, called the *canonical basis*. Denoting $n = n_1 - n_2$, it is given by

$$\left\{ \varphi^{\chi}_{j,m} = \sqrt{2j+1}\, D^{j}_{\frac{n}{2}m} \;\middle|\; j \geq \left|\frac{n}{2}\right|, |m| \leq j \right\} , \tag{D.8}$$

and consequently we have the decomposition

$$H_\chi = \bigoplus_{j=|\frac{n}{2}|}^{\infty} \mathcal{H}^{\chi}_j \,, \tag{D.9}$$

with $\mathcal{H}^{\chi}_j$ the Hilbert space spanned by the $(2j+1)$ basis elements. It is also common to define the Wigner $SL(2,\mathbb{C})$ matrices as

$$\begin{aligned} D^{\chi}_{j_1 m_1 j_2 m_2}(g) &= \langle \chi; j_1, m_1 |\, \rho^{\chi}(g)\, | \chi; j_2, m_2 \rangle \\ &= \int_{SU(2)} \mathrm{d}u \; (\varphi^*)^{\chi}_{j_1,m_1}(u) \rho^{\chi}(g) \varphi^{\chi}_{j_2,m_2}(u) \end{aligned} \tag{D.10}$$

and denote the elements of H_χ with the braket notation $|\chi; j, m\rangle$.

D.2 Harmonic analysis

D.2.1 Fourier transform on $L^2(SL(2,\mathbb{C}))$

We would like an analog of the usual Fourier transform on the square integrable functions on the group, defined with respect to the Haar measure. Although $SL(2,\mathbb{C})$ is not compact, it is locally compact and unimodular, so it has a bi-invariant measure.

Now, in the same way that the factor $e^{i\lambda x}$ that appears in the usual Fourier transform is the solution to the equation $f(x+y) = f(x)f(y)$, we will use the above representations on D_χ for the analog coefficient for functions on $L^2(SL(2,\mathbb{C}))$, since they satisfy the representation property $\rho^{\chi}(gh) = \rho^{\chi}(g)\rho^{\chi}(h)$. We then define the *Fourier transform* of $f \in L^2(SL(2,\mathbb{C}))$ to be

$$\hat{f}(\chi) = \int_{SL(2,\mathbb{C})} \mathrm{d}g \; f(g) \rho^{\chi}(g) \,. \tag{D.11}$$

It turns out, however, that this integral is only well-defined for the representations with $n_2 = -n_1^*$. In this case we may reparametrize

$$n_1 = \frac{1}{2}(n+ip) \qquad n_2 = \frac{1}{2}(-n+ip) \,, \tag{D.12}$$

with $n \in \mathbb{Z}$ and $\rho \in \mathbb{R}$, and we may equivalently label the representations with $\chi = (n,p)$. These are precisely the representations of the principal series mentioned above. The Fourier transform is then an operator on H_χ acting as

$$\hat{f}(\chi)\phi(u) = \int_{SL(2,\mathbb{C})} \mathrm{d}g \; f(g) \rho^{\chi}(g) \phi(u) \,. \tag{D.13}$$

It is possible to show that the action of this operator on the $SU(2)$ functions ϕ can be written in terms of an *integral kernel* $K(u,v;\chi)$ as

$$\hat{f}(\chi)\phi(u) = \int_{SU(2)} \mathrm{d}v\; K(u,v;\chi)\phi(v)\,. \tag{D.14}$$

An explicit formula for the kernel exists, and can be found in [57]. The kernel K turns out to be smooth in its arguments u, v, and entire analytic in p. Furthermore, if $K(u,v;\chi)$ is the kernel of $f(g)$, it satisfies the relations:

1. $f(gh)$ has kernel $\rho^{-\chi}(h)K(u,v;\chi)$,
2. $f(h^{-1}g)$ has kernel $\rho^{\chi}(h)K(u,v;\chi)$,
3. $f(g^{-1})$ has kernel $K(u,v;-\chi),\ -\chi = (-n_1,-n_2)$,
4. $f^*(g)$ has kernel $K(u,v;\chi^*)$,
5. $f^*(g^{-1})$ has kernel $K(u,v;-\chi^*)$.

Finally, we also need also define an appropriate trace for the operators $\hat{f}(\chi)$. This is given by the integral

$$\mathrm{Tr}[\hat{f}(\chi)] = \int_{SU(2)} \mathrm{d}u\; K(u,u,\chi)\,, \tag{D.15}$$

which can be shown to be well-defined for all $f \in L^2(SL(2,\mathbb{C}))$. This integral is called a trace because it agrees with the expectation that it should be a sum over matrix elements:

$$\begin{aligned}
\mathrm{Tr}[\hat{f}(\chi)] &= \sum_{j,m} \langle \chi; j,m|\, \hat{f}(\chi)\, |\chi; j,m\rangle \\
&= \sum_{j,m} \int_{SU(2)} \mathrm{d}u\,\mathrm{d}v\; (\varphi^*)^{\chi}_{j,m}(u) K(u,v;\chi) \varphi^{\chi}_{j,m}(v) \\
&= \int_{SU(2)} \mathrm{d}u\,\mathrm{d}v\; K(u,v;\chi)\delta(u-v) \\
&= \int_{SU(2)} \mathrm{d}u\; K(u,u;\chi)\,,
\end{aligned}$$

where we used the orthogonality relation of the canonical basis. The trace can furthermore be written in terms of the Wigner matrices of equation (D.10) as

$$\begin{aligned}
\mathrm{Tr}[\hat{f}(\chi)] &= \sum_{j,m} \langle \chi; j,m|\, \hat{f}(\chi)\, |\chi; j,m\rangle \\
&= \sum_{j,m} \langle \chi; j,m| \int_{SL(2,\mathbb{C})} \mathrm{d}g\; f(g)\rho^{\chi}(g)\, |\chi; j,m\rangle \\
&= \sum_{j,m} \int_{SL(2,\mathbb{C})} \mathrm{d}g\; f(g) D^{\chi}_{jmjm}(g)\,.
\end{aligned}$$

D.2.2 Plancherel Theorem

It turns out that the Fourier transformation of equation (D.11) has an inverse, given by the formula

$$\begin{aligned} f(g) &= \frac{1}{2}\int \mathrm{d}\chi\ (n^2+p^2)\mathrm{Tr}[\rho^{-\chi}(g)\hat{f}(\chi)] \\ &= \frac{1}{2}\sum_n \int_{-\infty}^{\infty} \mathrm{d}p\ (n^2+p^2)\int_{SU(2)} \mathrm{d}u\ \rho^{-\chi}(g)K(u,u;\chi)\,, \end{aligned}$$

where by $\mathrm{d}\chi$ we mean a sum over every integer n and an integral over every real p. The trace in this last formula can be rewritten in terms of the Wigner matrices by using the right invariance of the measure. Note, from the properties of the kernel, that if $K(u,v;\chi)$ is the kernel of $f(h)$, then $\rho^{-\chi}(g)K(u,v;\chi)$ is the kernel of $f(hg)$. Hence,

$$\begin{aligned} \mathrm{Tr}[\rho^{-\chi}(g)\hat{f}(\chi)] &= \sum_{j,m}\int_{SL(2,\mathbb{C})} \mathrm{d}h\ f(hg)D^{\chi}_{j,m,j,m}(h) \\ &= \sum_{j,m}\sum_{l,q}(D^*)^{\chi}_{jmlq}(g)\int_{SL(2,\mathbb{C})} \mathrm{d}h\ f(h)D^{\chi}_{jmlq}(h)\,, \end{aligned}$$

and the inverse transformation becomes

$$f(g) = \frac{1}{2}\sum_{j,m,l,q}\int \mathrm{d}\chi\ (n^2+p^2)\int_{SL(2,\mathbb{C})} \mathrm{d}h\ (D^*)^{\chi}_{jmlq}(g)f(h)D^{\chi}_{jmlq}(h)\,. \tag{D.16}$$

One can moreover show that an analogue of Parseval's identity holds,

$$\langle f_1|f_2\rangle = \langle K_1|K_2\rangle\ , \tag{D.17}$$

where K_1, K_2 are the integral kernels of f_1, f_2, respectively. This result establishes a correspondence between integration over functions in $SL(2,\mathbb{C})$ and integration over kernels in $SU(2)$.

D.2.3 Decomposition of the regular representation

As we did in Appendix B for the case of compact groups, we consider now the right regular representation on the square integrable functions, defined as

$$\begin{aligned} R: SL(2,\mathbb{C}) &\to U\left[L^2(SL(2,\mathbb{C}))\right] \\ h &\mapsto \big(R_h: f(g)\mapsto f(gh)\big)\,, \end{aligned} \tag{D.18}$$

which is unitary because of the bi-invariance of the Haar measure. The expression found above for the Fourier transformation gives rise to a decomposition of the regular representation in terms of a *direct integral* of Hilbert spaces (as opposed to the discrete direct sum case). It is not necessary to understand the construction of the direct integral of Hilbert spaces; it suffices to have the concept of a continuous labeling of Hilbert spaces $H(x)$, and denote the full Hilbert space by $\int_{\oplus} \mathrm{d}\mu\, H(x)$, with μ a measure on the space of continuous labels x. Consider then the decomposition of H_{χ} into invariant subspaces as in equation (D.9). We construct the Hilbert space of kernels using the principal series

$$\int_{\oplus} \mathrm{d}\chi \bigoplus_{j=|\frac{n}{2}|}^{\infty} \mathcal{H}_j^{\chi} \simeq L^2(SL(2,\mathbb{C}))\,, \tag{D.19}$$

which must be isomorphic to $L^2(SL(2,\mathbb{C}))$ through the Plancherel theorem. A right translation $f(g) \mapsto f(gh)$ induces a transformation on the kernel as $K(u,v;\chi) \mapsto \rho^{-\chi}(h)K(u,v;\chi)$, implying that this is indeed a decomposition into irreducible unitary representations.

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Declaration of Authorship

The author hereby declares that this Master's Thesis has been written solely by himself and is the fruit of his own work. All the content not directly produced by the author has been acknowledged as such and referenced in the bibliography. This work was not submitted elsewhere for any other purposes.

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